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On Pitman domination

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Iowa State University, 1993

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300 N. Zeeb Rd.
Ann Arbor, MI 48106

On Pitman domination

by

Seongmo Yoo

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of the
Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Department: Statistics
Major: Statistics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa
1993

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CHAPTER 1. INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

In statistics, we frequently encounter estimation problems. Such problems call for deciding on an estimation method. Such estimation methods include the method of moments, maximum likelihood, least squares, minimum variance unbiased estimation, minimum chi-square, and minimum distance, among others. These estimation methods usually arise in connection with formulating certain “optimum” properties, followed by the task of finding estimators that are optimum within given classes. Some of these optimum properties involve ideas of bias, variance, efficiency and consistency. Thus, most estimation methods are to be viewed in the context of a given optimum property and given class.

All of the above-mentioned criteria are marginal in nature, in that the optimality or near-optimality of an estimator can be ascertained for that estimator itself, without introduction, generally, of other members of the pertinent class.

A slightly different point of view, emphasizing the idea of comparison of estimators, was proposed by Pitman [26]: an estimator X is closer than Y to a scalar parameter θ if

$$\Pr_{\theta}(|X - \theta| < |Y - \theta|) > 1/2 \quad , \quad \forall \quad \theta .$$

This criterion is now called Pitman Closeness Criterion (PCC) or Pitman Nearness

Criterion (PNC) [27]. It is generalized by Rao [28] and Sen [32] in a manner that may be formulated as follows: Let the p -dimensional vectors X and Y , with joint density depending on a parameter vector $\omega = (\theta_1, \dots, \theta_p, \omega_1, \dots, \omega_q)$, be estimators of $\theta = (\theta_1, \dots, \theta_p)$. X is closer than Y to θ with respect to the loss function $L(\cdot, \cdot)$ if

$$\Pr_{\omega}(L(X, \theta) < L(Y, \theta)) > 1/2, \quad \forall \omega \in \Omega. \quad (1.1)$$

This criterion is now called the Generalized Pitman Closeness Criterion (GPCC) or Generalized Pitman Nearness Criterion (GPNC).

Pitman [26] suggested that median-unbiased estimators derived from sufficient statistics are well suited to PCC, and gave the “comparison theorem” for identifying classes of estimators Pitman-dominated by a median-unbiased estimator. He also noted that PCC is intransitive.

After Pitman gave the “comparison theorem” for identifying classes of estimators Pitman-dominated by median-unbiased estimators derived from sufficient statistics, Ghosh and Sen [10] and Nayak [22] showed that a median-unbiased estimator is best equivariant in the Pitman sense. These investigations are in a sense supportive of Pitman’s idea.

Following a different line of research based on certain shrinkage constructions, Salem and David [31] constructed a shrinkage estimator for θ Pitman-dominating an observation X from a normal population with unknown mean θ and unit variance. (see also Efron [7] for an example in a similar vein). David and Salem [5] extended the results of Salem and David [31] to the case of a single observation from any symmetric density, and also constructed intransitive triples of estimators of a Laplace location parameter, each member of the triple Pitman-dominating a single observation. This direction of research is less supportive of Pitman’s idea.

In this dissertation, we generalize the approach of David and Salem [5]. A number of parametric situations are considered, including some considered by Pitman. In each case, a class of continuous not necessarily increasing functions of a median-unbiased or otherwise natural estimator derived from sufficient statistics is considered, each member of the class Pitman-dominating the estimator itself. Special attention is given to Pitman domination for location-scale families. Finally, we construct Pitman-intransitive triples of estimators based on the earlier results on shrinkage and equivariant estimators.

Section 1.2 is devoted to further details concerning the pertinent literature.

Chapter 2 is mainly devoted to univariate shrinkage constructions for dominating an estimator in the sense of PCC and GPCC. In Section 2.2 we discuss the relation between median-unbiased estimators and PCC. In Section 2.3 we study construction of estimators Pitman-dominating competing, in particular sufficient median-unbiased, estimators.

Chapter 3 is devoted to univariate shrinkage constructions for location-scale families of density functions. In Section 3.2 we study shrinkage of a single observation when the scale parameter is bounded below. In Section 3.3 we show that, in a certain sense, the lower bound on the scale parameter cannot be removed.

Chapter 4 is devoted to multivariate shrinkage constructions for Pitman domination. In Section 4.2 we review the results on James-Stein type estimators for Pitman domination. In Section 4.3 we adapt univariate Pitman domination to multivariate and generalized Pitman domination problems.

Chapter 5 is devoted to the constructions of triples of intransitive estimators for location parameters and scale parameters, respectively in Section 5.2 and Section 5.3.

1.2 Literature Review

After Pitman proposed the closeness criterion, research proceeded in many directions. One direction was to compare the Pitman closeness criterion with other estimation criteria; another direction was to generalize the closeness criterion; still another direction was to find families of estimators Pitman-dominated by a median-unbiased estimator. A somewhat different direction was to find families of estimators Pitman-dominates a median-unbiased estimator. We review the literatures regarding these directions in this section.

1.2.1 The Pitman Closeness Criterion as an Estimation Criterion

The arguments concerning which estimation method should be used for comparing estimators of a parameter have to some extent been subjective, touching on what optimum property is regarded as critical. After Pitman proposed the closeness criterion, it has been compared with other criteria.

Geary [9] discussed the relation between the Pitman closeness criterion and the criterion of minimum variance for unbiased estimators. He also noted that the Pitman closeness criterion is identical with the criterion of efficiency as determined by a comparison of the variances of two estimators when the joint distribution of the estimators is normal and that the Pitman criterion will not yield results much different from results based on efficiency when the estimators are based on large samples and are consistent.

Johnson [14] compared the Pitman closeness criterion with the mean square error criterion in some examples, and stated that comparison of mean square errors appeared to be somewhat more satisfactory than the Pitman closeness criterion. On

the contrary, Rao [27] compared the Pitman closeness criterion and mean square error criterion by providing examples in which shrinking an unbiased estimator to a minimum mean square error estimator did not improve the property of closeness in Pitman sense, and suggested that the Pitman closeness criterion could be considered as a criterion to compare estimators.

Peddada [23] showed that the Pitman closeness criterion, minimum mean squared criterion, and minimum mean absolute error criterion are equivalent under some conditions on the moments of random variable.

The Pitman closeness criterion is generalized by Rao *et al.* [28] and Sen *et al.* [32] with respect to the loss function $L(\cdot, \cdot)$ as follows: Let the p -dimensional vectors X and Y , with joint density depending on a parameter vector $\omega = (\theta_1, \dots, \theta_p, \omega_1, \dots, \omega_q)$, be estimators of $\theta = (\theta_1, \dots, \theta_p)$. X is closer than Y to θ with respect to the loss function $L(\cdot, \cdot)$ if

$$\Pr_{\omega}(L(X, \theta) < L(Y, \theta)) > 1/2, \quad \forall \omega \in \Omega. \quad (1.2)$$

This criterion is now called the Generalized Pitman Closeness Criterion (GPCC) or Generalized Pitman Nearness Criterion (GPNC).

According to Hwang [12] X stochastically dominates Y under loss function L if, for any real number c ,

$$\Pr_{\theta}(L(X, \theta) \geq c) \leq \Pr_{\theta}(L(Y, \theta) \geq c), \quad \forall \theta$$

and the probability inequality is strict for some θ . Lee [21] showed that stochastic domination implies Pitman domination for comparing any two estimators of the mean vector of a multivariate normal distribution if the distribution of estimator is multivariate normal, and also showed that stochastic domination and Pitman domination

are equivalent if the two estimators are unbiased.

1.2.2 Pitman Domination of Equivariant Estimators

Pitman [26] suggested that a median-unbiased estimator based on a minimal sufficient statistic should fare well under his closeness criterion.

After that, Basu [1] defined an ancillary statistic and showed that a complete sufficient statistic is independent of any ancillary statistic.

Ghosh and Sen [10], interpreting Pitman's comparison theorem in terms of Basu's theorem [1], showed that a median-unbiased estimator dominates every other estimator within the class of equivariant estimators. Nayak [22] obtained best equivariant estimators in the sense of GPCC by using decision theoretic approaches. Kubokawa [19] showed that an estimator is median-unbiased if and only if it is best equivariant in the sense of GPCC.

1.2.3 Pitman-Dominating Median-Unbiased Estimators

Salem and David [31] and Salem [30] constructed a shrinkage estimator for θ Pitman-dominating an observation X from a normal population with unknown mean θ and unit variance. They constructed a continuous increasing function $f(x)$, defined by

$$f(x) = \frac{.3x}{.89} \quad , \quad 0 \leq x \leq .89 \quad ,$$

and

$$f(x + R(x)/2) = x \quad , \quad .89 < x < \infty$$

with

$$R(x) = \Phi^{-1} \left(\frac{3}{2} - \Phi(x) \right) \quad ,$$

$\Phi(\cdot)$ the unit normal cdf. With $f(-|x|)$ defined to be $-f(|x|)$, they showed that $f(X)$ dominates X in the sense of PCC.

Also, Efron [7], using a similar construction, showed that $\delta(\bar{X})$ dominates \bar{X} in the sense of PCC, with $\delta(x)$ defined as follows: $\delta(x) = x - \Delta(x)$, for $x \geq 0$, with

$$\Delta(x) = \frac{1}{2\sqrt{n}} \min(\sqrt{n} x, \Phi(-\sqrt{n} x))$$

and, again, $\delta(-|x|) = -\delta(|x|)$.

Given any location family symmetric about a location parameter θ , David and Salem [5] provided a way to construct a boundary function $\mu(x)$ such that, for any continuous increasing $T(x)$ between x and $\mu(x)$, $T(X)$ dominates X in the sense of PCC.

1.2.4 Multivariate Normal Considerations

Consider the observation vector $X = (X_1, \dots, X_p)'$ where $X \sim N_p(\theta, V)$ for $p \geq 3$. In the case of $V = \sigma^2 I$, James and Stein [13] proved that T , where

$$T = (1 - c/X'X)X$$

has smaller mean squared error than X for all θ provided $0 < c < 2(p - 2)$ and the mean squared error is minimized when $c = p - 2$.

Consider a loss function of the form $L(x, \theta) = (X - \theta)'(X - \theta)$. Efron [7] stated that James-Stein estimator [13] Pitman-dominates the vector of sample means with respect to the loss function $L(\cdot, \cdot)$ in estimating the mean vector of the normal density with $V = \sigma^2 I$.

Rao, Keating, and Mason [28] numerically showed that the James-Stein estimator dominates the vector of sample means in the Pitman sense. Keating and Mason

[17] proposed an alternative James-Stein estimator with $c = p - 1$, and numerically showed that it dominates the James-Stein estimator in the Pitman sense.

Sen, Kubokawa, and Saleh [32] showed that, for $p \geq 2$, any Stein type estimator with c , $0 < c < (p - 1)(3p + 1)/(2p)$, dominates the vector of sample means in the Pitman sense.

1.2.5 Scale Parameter Estimation

Let X_1, \dots, X_n be a random sample of size n from normal density with mean μ and variance σ^2 . Some general aspects of estimating σ^2 are as follows.

Eisenhart [6] noted that the square roots of the usual estimators of σ^2 , such as the sufficient unbiased estimator, the maximum likelihood estimator, and the minimum mean absolute error estimator of σ^2 , tend to underestimate σ in the probability sense.

Keating [16] showed that the sufficient median-unbiased estimator of σ is Pitman-closest in the class of estimators of the form αS .

Nayak [22], in the context of his paper on PCC, pointed out that the *log* transformation will transform a scale parameter problem into a location parameter problem.

Khattree [18] noted that maximum likelihood estimator under the unknown mean assumption has smaller mean square error than that under the known mean assumption in the problem of estimating variance σ^2 with known mean μ , and thus compared the following two classes of estimators σ^2 in the Pitman sense:

$$C_1 : \{c_1 S_1 : 0 < c_1 \leq 1\}$$

and

$$C_2 : \{c_1 S_2 : 0 < c_1 \leq 1\} ,$$

where $S_1 = \sum_{i=1}^n (X_i - \mu)^2$, $S_2 = \sum_{i=1}^n (X_i - \bar{X})^2$.

CHAPTER 2. UNIVARIATE SHRINKAGE CONSTRUCTIONS

2.1 Introduction

In section 2.2, we study the relation between median-unbiased estimators and PCC in some detail. In section 2.3, we essentially generalize the approach of David and Salem [5]. A number of parametric situations are considered, including some considered by Pitman. In each case, a class of continuous not necessarily increasing functions of a minimal sufficient median-unbiased or otherwise natural estimator is considered, each member of the class Pitman-dominating the estimator itself. Three examples are considered:

1. The parameter of interest is the median of a density supported by the real line. The estimator to be dominated is a single observation, minimal sufficient median-unbiased for the population median. This is the single-observation location parameter situation.
2. The parameter of interest is the non-centrality parameter of a non-central t distribution. The estimator to be dominated is $\sqrt{n}\bar{X}/S$, for sample sizes two and three.
3. The parameter of interest is a positive parameter of a density supported by the positive real line. The estimator to be dominated is median-unbiased for the

parameter. This example includes scale parameter situations.

2.2 Median-Unbiased Estimators and the Pitman Closeness Criterion

An absolutely continuous estimator X of a unknown parameter θ is said to be median-unbiased (or, θ is the median of the density of X) if

$$\Pr_{\theta}(X \leq \theta) = \Pr_{\theta}(X \geq \theta) , \quad \forall \theta .$$

Pitman [26] gave the following theorem to identify classes of estimators dominated in the Pitman sense by a median-unbiased estimator.

Theorem 2.1 (Pitman Comparison Theorem) *Let X_1 be an absolutely continuous random variable with median value θ , and let X_2 be any other continuous random variable; then X_1 is a closer estimate of θ than X_2 if there exist a random variable Z , always of one sign, and such that*

$$X_1 \quad \text{and} \quad Z(X_2 - X_1)$$

are independent. That is,

$$\Pr_{\theta}(|X_1 - \theta| < |X_2 - \theta|) > \frac{1}{2} , \quad \forall \theta .$$

Proof: The inequality $|X_1 - \theta| < |X_2 - \theta|$ is satisfied if any one of the following is true:

$$(A) \quad X_1 > \theta , \quad X_1 < X_2$$

$$(B) \quad X_1 > \theta \quad , \quad X_1 + X_2 < 2\theta$$

$$(C) \quad X_1 < \theta \quad , \quad X_1 > X_2$$

$$(D) \quad X_1 < \theta \quad , \quad X_1 + X_2 > 2\theta \quad .$$

Without loss of generality, suppose Z is positive. Let p be the probability that $Z(X_2 - X_1) > 0$, i.e., $X_2 > X_1$. Since $Z(X_2 - X_1)$ is independent of X_1 , $\Pr_\theta((A)) = \frac{1}{2}p$ and $\Pr_\theta((C)) = \frac{1}{2}(1 - p)$. Hence,

$$\begin{aligned} \Pr_\theta(|X_1 - \theta| < |X_2 - \theta|) &= \Pr_\theta((A)) + \Pr_\theta((B)) + \Pr_\theta((C)) + \Pr_\theta((D)) \\ &> \Pr_\theta((A)) + \Pr_\theta((C)) \\ &= \frac{1}{2}p + \frac{1}{2}(1 - p) \\ &= \frac{1}{2} \end{aligned} \quad (\text{Q.E.D.})$$

This proof is essentially given by Pitman [26]. With this comparison theorem, Pitman suggested that a median-unbiased estimator depending on a minimal sufficient statistic is well suited to PCC.

Basu [1] defined an ancillary statistic and showed that a complete sufficient statistic is independent of any ancillary statistic. It can be noted that the statistic $Z(X_2 - X_1)$ in Theorem 2.1 involves an ancillary statistic if we let X_1 be complete sufficient and X_2 be the sum of X_1 and an ancillary statistic.

Let X_1, \dots, X_n be a random sample of size n from normal density with mean μ known and variance σ^2 unknown. Consider estimating variance σ^2 . Let S^2 be the unbiased estimator of σ^2 (i.e., $S^2 = \sum_{i=1}^n (X_i - \mu)^2 / n$). The following proposition is noted by Pitman [26].

Proposition 2.1 (Pitman) For given $n \geq 1$, $\Pr(0 < \chi_n^2 < n) > \frac{1}{2}$.

We note that Proposition 2.1 says that

$$\begin{aligned} \Pr(0 < \chi_n^2 < n) &= \Pr_{\sigma}(0 < \frac{nS^2}{\sigma^2} < n) \\ &= \Pr_{\sigma}(0 < S^2 < \sigma^2) \\ &> \frac{1}{2} . \end{aligned}$$

This implies that the unbiased estimator of σ^2 tends to underestimate its value as noted by Eisenhart [6]. Thus we need improve this unbiased estimator S^2 in order not to underestimate.

Keating [16] considered the estimation of scale parameters in location and scale parameter families of density functions. Let T be a scale invariant median-unbiased estimator of a scale parameter σ . Consider a class S_1 of estimators of σ such that

$$S_1 = \{cT : c > 0\} .$$

If T is a complete and sufficient statistic for σ then S_1 contains the maximum likelihood estimator and minimum variance unbiased estimator of σ .

Theorem 2.2 (Keating) Let T be a scale invariant median-unbiased estimator of a scale parameter σ in a location and scale parameter family of density functions. Let cT be an estimator in S_1 such that $c \neq 1$. Then, for every $c > 0$, $c \neq 1$, a scale invariant median-unbiased estimator T is Pitman-closest in class S_1 , i.e.,

$$\Pr_{\sigma}(|T - \sigma| < |cT - \sigma|) > \frac{1}{2} , \quad \forall \sigma .$$

Proof: When $0 < c < 1$, we have that

$$\begin{aligned} \Pr_{\sigma}(|T - \sigma^2| < |cT - \sigma^2|) &= \Pr_{\sigma}(T < \frac{2}{1+c}\sigma^2) \\ &> \Pr_{\sigma}(T < \sigma^2) = \frac{1}{2} . \end{aligned}$$

Similarly, when $c > 1$, we have that

$$\begin{aligned} \Pr_{\sigma}(|T - \sigma^2| < |cT - \sigma^2|) &= \Pr_{\sigma}(T > \frac{2}{1+c}\sigma^2) \\ &> \Pr_{\sigma}(T > \sigma^2) = \frac{1}{2} . \end{aligned} \quad (\text{Q.E.D.})$$

Example 2.1 Let X_1, \dots, X_n be a random sample of size n from normal density with mean unknown μ and variance σ^2 . Consider the problem of estimation of the variance σ^2 . Let S^2 be the unbiased estimator of σ^2 (i.e., $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2/n$). Let S_{MU}^2 be the median-unbiased estimator of σ^2 . Then, in view of Proposition 2.1, there exist $k_n, 0 < k_n < 1$, such that

$$\Pr_{\sigma}(0 < \frac{S^2}{k_n} < \sigma^2) = \frac{1}{2} .$$

We have $S_{MU}^2 = \frac{S^2}{k_n} > S^2$. Therefore, by Theorem 2.2, S_{MU}^2 is Pitman-closer to σ^2 than S^2 . \square

Example 2.2 Let X_1, \dots, X_n be a random sample of size n from uniform density $(0, \theta)$, i.e., $X_i \sim U(0, \theta)$. Consider the problem of estimation of θ . Let $X_{(n)}$ be the largest observation from the sample. Note that $X_{(n)}$ is a maximum likelihood estimator, $\frac{n+1}{n}X_{(n)}$ is a uniformly minimum variance unbiased estimator, $\frac{n+2}{n+1}X_{(n)}$ is a

estimator minimizing mean square error. Then, by Theorem 2.2, a median-unbiased estimator $2^{1/n}X_{(n)}$ is Pitman-closer than any of them. \square

Ghosh and Sen [10], and Nayak [22] restricted the classes of estimators Pitman-dominated by an appropriate median-unbiased estimator in light of Basu's theorem [1]. They showed a median-unbiased estimator based on complete sufficient statistic of a parameter is the Pitman-closest estimator within a class of equivariant estimators satisfying some conditions. The following two theorems are given by Ghosh and Sen [10].

Theorem 2.3 (Ghosh and Sen) *Let T be a median-unbiased estimator of θ , and consider the class \mathcal{C} of all statistics of the form $U = T + Z$, where T and Z are independently distributed. Then*

$$\Pr_{\theta}(|T - \theta| < |U - \theta|) > \frac{1}{2}, \quad \forall \theta, \quad \forall U \in \mathcal{C}.$$

Proof: Note that

$$\begin{aligned} & \Pr_{\theta}(|T - \theta| > |U - \theta|) \\ &= \Pr_{\theta}((T - \theta)^2 - \{(T - \theta) + Z\}^2 > 0) \\ &= \Pr_{\theta}(2Z(T - \theta) < -Z^2) \\ &< \Pr_{\theta}(Z(T - \theta) < 0) \\ &= \Pr_{\theta}(Z(T - \theta) < 0 | T > \theta) \left(\frac{1}{2}\right) + \Pr_{\theta}(Z(T - \theta) < 0 | T < \theta) \left(\frac{1}{2}\right) \\ &= \frac{1}{2} \{ \Pr_{\theta}(Z < 0 | T > \theta) + \Pr_{\theta}(Z > 0 | T < \theta) \} \\ &= \frac{1}{2} \{ \Pr_{\theta}(Z > 0) + \Pr_{\theta}(Z < 0) \} \end{aligned}$$

$$= \frac{1}{2}$$

where the last second is due to the independence of Z and T . (Q.E.D.)

Theorem 2.4 (Ghosh and Sen) *Let \mathcal{C}^* be the class of all statistics of the form $U = T(1 + Z)$, where T is median-unbiased estimator of θ and nonnegative and T and Z are independently distributed. Then*

$$\Pr_{\theta}(|T - \theta| < |U - \theta|) > \frac{1}{2}, \quad \forall \theta, \forall U \in \mathcal{C}^*.$$

Proof: By noting that T is nonnegative, we have that

$$\begin{aligned} & \Pr_{\theta}(|T - \theta| > |U - \theta|) \\ &= \Pr_{\theta}((T - \theta)^2 - \{(T - \theta) + TZ\}^2 > 0) \\ &= \Pr_{\theta}(2TZ(T - \theta) < -T^2Z^2) \\ &< \Pr_{\theta}(Z(T - \theta) < 0) \\ &= \Pr_{\theta}(Z(T - \theta) < 0 | T > \theta) \left(\frac{1}{2}\right) + \Pr_{\theta}(Z(T - \theta) < 0 | T < \theta) \left(\frac{1}{2}\right) \\ &= \frac{1}{2} \{ \Pr_{\theta}(Z < 0 | T > \theta) + \Pr_{\theta}(Z > 0 | T < \theta) \} \\ &= \frac{1}{2} \{ \Pr_{\theta}(Z > 0) + \Pr_{\theta}(Z < 0) \} \\ &= \frac{1}{2} \end{aligned}$$

where the last second is due to the independence of Z and T . (Q.E.D.)

2.3 Construction of Estimators Pitman-Dominating Competing Estimators

2.3.1 Characterization of Pitman Domination

Before we consider characterization for Pitman estimators, we first define some notations and a terminology which were first defined by Rao, Keating, and Mason [28]. Let T_1 and T_2 be two univariate estimators of the real parameter θ . We further define

$$\begin{aligned}\Omega_1 &= \{(T_1, T_2) : T_1 + T_2 > 2\theta \text{ and } T_1 < T_2\} \\ \Omega_2 &= \{(T_1, T_2) : T_1 + T_2 < 2\theta \text{ and } T_1 < T_2\} \\ \Omega_3 &= \{(T_1, T_2) : T_1 + T_2 < 2\theta \text{ and } T_2 < T_1\} \\ \Omega_4 &= \{(T_1, T_2) : T_1 + T_2 > 2\theta \text{ and } T_2 < T_1\} .\end{aligned}$$

Definition 2.1 (Rao, Keating and Mason) *The point x in the essential range of X is said to be a crossing point of T_1 and T_2 if $T_1 - T_2$ changes sign at x (i.e., for given $\epsilon > 0$, $(T_1(x - \epsilon) - T_2(x - \epsilon))(T_1(x + \epsilon) - T_2(x + \epsilon)) < 0$).*

In general, the crossing points of T_1 and T_2 will just consist of the points of intersection of T_1 and T_2 .

In an effort to characterize Pitman domination, Rao, *et al.* [28] stated and proved following theorem.

Theorem 2.5 (Rao, Keating and Mason)

$$\Pr_{\theta}(|T_1 - \theta| < |T_2 - \theta|) = \Pr_{\theta}(\Omega_1) + \Pr_{\theta}(\Omega_3) .$$

Proof: In Ω_1 , we have that $T_1 < T_2$ which implies that $T_1 - \theta < T_2 - \theta$. And for each $(T_1, T_2) \in \Omega_1$, we have that

$$T_1 + T_2 > 2\theta \quad \text{and} \quad -T_1 + T_2 > 0 .$$

By adding these two inequalities, we obtain $T_2 > \theta$. Therefore, we have that,

$$-(T_2 - \theta) < (T_1 - \theta) < (T_2 - \theta) .$$

In Ω_3 , we have that $T_1 > T_2$ which implies that $T_1 - \theta > T_2 - \theta$. And for each $(T_1, T_2) \in \Omega_3$, we have that

$$T_1 + T_2 < 2\theta \quad \text{and} \quad -T_1 + T_2 < 0 .$$

By adding these two inequalities, we obtain $T_2 < \theta$. Therefore, we have that,

$$-(\theta - T_2) < (T_1 - \theta) < (\theta - T_2) .$$

In a similar way we can show that in Ω_2 and Ω_4 ,

$$|T_2 - \theta| < |T_1 - \theta| .$$

Hence it is now obvious that

$$\Pr_{\theta}(|T_1 - \theta| < |T_2 - \theta|) = \Pr_{\theta}(\Omega_1) + \Pr_{\theta}(\Omega_3) . \quad (\text{Q.E.D.})$$

It can be noted that Theorem 2.5 emphasizes comparison of estimators. Suppose we are interested in shrinkage constructions of estimators $T \equiv T(X)$ Pitman-dominating X . We can interpret Theorem 2.5 in terms of construction of Pitman-dominating estimators in the following way. For simplicity, we consider there is at

most one crossing point of T and X . Assume X is a median-unbiased estimator of θ , and $T(X)$ is continuous and increasing in X . Then, without loss of generality, there are the following four possible cases:

$$(i) \quad T > X$$

$$(ii) \quad T < X$$

$$(iii) \quad T < X < 0, T(0) = 0, 0 < X < T$$

$$(iv) \quad X < T < 0, T(0) = 0, 0 < T < X$$

We now establish that, if a T is to be found, of either type (i), (ii), (iii) or (iv), Pitman-dominating X , then such a T must be of type (iv).

To begin with, in view of Theorem 2.5, we have that

$$\begin{aligned} & \Pr_{\theta}(|T - \theta| < |X - \theta|) \\ &= \Pr_{\theta}(T + X > 2\theta, T < X) + \Pr_{\theta}(T + X < 2\theta, T > X) . \end{aligned}$$

In the case of (i) $T > X$, we have that

$$\begin{aligned} & \Pr_{\theta}(|T - \theta| < |X - \theta|) \\ &= \Pr_{\theta}(T + X > 2\theta, T > X) \\ &< \Pr_{\theta}(X > \theta) = \frac{1}{2} , \end{aligned}$$

so that a T Pitman-dominating X cannot exist.

In the case of (ii) $T < X$, we have that

$$\Pr_{\theta}(|T - \theta| < |X - \theta|)$$

$$\begin{aligned}
&= \Pr_{\theta}(T + X < 2\theta \ , \ T < X) \\
&< \Pr_{\theta}(X < \theta) = \frac{1}{2} \ ,
\end{aligned}$$

so that, again, a T Pitman-dominating X cannot be found.

In the case of (iii) $T < X < 0$, $T(0) = 0$, $0 < X < T$, for given $\theta > 0$, we have that

$$\begin{aligned}
&\Pr_{\theta}(|T - \theta| < |X - \theta|) \\
&= \Pr_{\theta}(T + X < 2\theta \ , \ T > X > 0) \\
&< \Pr_{\theta}(0 < X < \theta) < \frac{1}{2} \ ,
\end{aligned}$$

so that, again, a T Pitman-dominating X cannot exist.

Thus, in the case of (i), (ii), and (iii), no T can be found Pitman-dominating X .

Finally, in the case of (iv) $X < T < 0$, $T(0) = 0$, $0 < T < X$, now define $\phi \equiv \phi(X)$, T , and a continuous increasing function $g^{-1}(X)$ as follows:

$$\begin{aligned}
0 &< \phi(X) < X \ , \ X > 0 \\
X &< \phi(X) < 0 \ , \ X < 0
\end{aligned}$$

and

$$T \equiv X - \phi$$

and

$$g^{-1}(X) = X - \frac{\phi}{2} \ .$$

Note that

$$\begin{aligned}
 T &= X - \phi \\
 &= 2(X - \frac{\phi}{2}) - X \\
 &= 2g^{-1}(X) - X .
 \end{aligned}$$

We may observe that

$$\begin{aligned}
 |T - \theta|^2 - |X - \theta|^2 &= |(X - \theta) - \phi|^2 - |X - \theta|^2 \\
 &= -2(X - \theta)\phi + \phi^2 .
 \end{aligned}$$

Thus

$$\begin{aligned}
 &\Pr_{\theta}(|T - \theta| < |X - \theta|) \\
 &= \Pr_{\theta}(|T - \theta|^2 < |X - \theta|^2) \\
 &= \Pr_{\theta}((X - \theta)\phi > \frac{\phi^2}{2}) \\
 &= \Pr_{\theta}((X - \theta)\phi > \frac{\phi^2}{2}, X > 0) + \Pr_{\theta}((X - \theta)\phi > \frac{\phi^2}{2}, X < 0) \\
 &= \Pr_{\theta}(X - \frac{\phi}{2} > \theta, X > 0) + \Pr_{\theta}(X - \frac{\phi}{2} < \theta, X < 0) .
 \end{aligned}$$

Suppose $\theta > 0$. Then

$$\begin{aligned}
 &\Pr_{\theta}(|T - \theta| < |X - \theta|) \\
 &= \Pr_{\theta}(X - \frac{\phi}{2} > \theta, X > 0) + \Pr_{\theta}(X - \frac{\phi}{2} < \theta, X < 0) \\
 &= \Pr_{\theta}(g^{-1}(X) > \theta) + \Pr_{\theta}(X < 0) \\
 &= \Pr_{\theta}(X > g(\theta)) + \Pr_{\theta}(X < 0) \\
 &= 1 - \Pr_{\theta}(0 < X < g(\theta)) .
 \end{aligned}$$

Suppose $\theta < 0$. Then

$$\begin{aligned}
& \Pr_{\theta}(|T - \theta| < |X - \theta|) \\
&= \Pr_{\theta}(X - \frac{\phi}{2} > \theta, X > 0) + \Pr_{\theta}(X - \frac{\phi}{2} < \theta, X < 0) \\
&= \Pr_{\theta}(X > 0) + \Pr_{\theta}(X - \frac{\phi}{2} < \theta) \\
&= \Pr_{\theta}(X > 0) + \Pr_{\theta}(X < g(\theta)) \\
&= 1 - \Pr_{\theta}(g(\theta) < X < 0) .
\end{aligned}$$

Thus, if there exists $g(\cdot)$ such that

$$\Pr_{\theta}(g(\theta) < X < 0) < \frac{1}{2} , \quad \theta < 0$$

and

$$\Pr_{\theta}(0 < X < g(\theta)) < \frac{1}{2} , \quad \theta > 0 ,$$

then T Pitman-dominates X . This result provides the motivation of the following section.

2.3.2 Shrinkage Constructions for Pitman Domination

The following lemma and two theorems in this section involve a version of certain continuous increasing function $\lambda(\cdot)$, each with its own particular domain, and a function

$$2\lambda^{-1}(x) - x \tag{2.1}$$

constructed from $\lambda(x)$. For ease of notation, the function (2.1) will be denoted throughout this section by $\mu_{\lambda}(x)$.

Lemma 2.1 *Consider a scalar statistic X , with density $f(x;\theta)$ depending on a scalar parameter θ , $0 \leq \theta < +\infty$, where, for all θ , the support of $f(x;\theta)$ is the nonnegative real line. Suppose that there is a continuous increasing function $\lambda(\theta)$, $\theta \geq 0$, such that*

$$\begin{aligned}\lambda(0) &= 0, \\ \lambda(\theta) &> \theta > 0,\end{aligned}\tag{2.2}$$

and

$$\int_0^{\lambda(\theta)} f(x;\theta) dx \leq \frac{1}{2}, \quad \theta \geq 0.\tag{2.3}$$

Define the continuous function $\mu(x)$:

$$\mu(x) = \max(0, \mu_\lambda(x)), \quad x > 0.\tag{2.4}$$

for which, in view of (2.1) and (2.2) with argument x ,

$$x > \mu(x), \quad x > 0.\tag{2.5}$$

Then any estimator $T(X)$ of θ , with $T(\cdot)$ continuous, and

$$\mu(x) < T(x) < x, \quad x > 0,\tag{2.6}$$

Pitman-dominates the estimator X ; in other words, for $\theta \geq 0$,

$$\Pr_\theta(|T(X) - \theta| < |X - \theta|) > \frac{1}{2}.\tag{2.7}$$

Proof: When $\theta = 0$, in view of (2.6), it is clear that

$$\Pr_\theta(T(X) < X) = 1$$

and thus (2.7) holds.

For $\theta > 0$, in view of (2.3), (2.7) is verified by verifying inequalities (2.14) and (2.24) below ; the first is verified in PART I, and the second in PART II.

PART I: Consider, for $\theta > 0$, any x such that

$$x \geq \lambda(\theta) . \quad (2.8)$$

Then, with regard particularly to the relative magnitudes of θ and $\mu(x)$, we have, in view of (2.2) and (2.5), the two mutually exclusive cases:

$$\theta \leq \mu(x) < x \quad (2.9)$$

and

$$\mu(x) < \theta < x . \quad (2.10)$$

Relation (2.9) clearly implies

$$|\mu(x) - \theta| < |x - \theta| . \quad (2.11)$$

As to relation (2.10), consider the relation

$$\theta \leq \lambda^{-1}(x) = \frac{x + \mu\lambda(x)}{2} \leq \frac{x + \mu(x)}{2} , \quad (2.12)$$

where the first weak inequality is due to (2.8), the equality is due to (2.1), and the second weak inequality is due to (2.4). Relation (2.10), together with (2.12), implies

$$0 < \theta - \mu(x) \leq x - \theta$$

and thus also implies

$$|\mu(x) - \theta| \leq |x - \theta| . \quad (2.13)$$

So, all told, (2.8) implies (2.13), and this implication allows writing

$$\int_{\lambda(\theta)}^{+\infty} f(x; \theta) dx \leq \int_{\{x : |\mu(x) - \theta| \leq |x - \theta|\}} f(x; \theta) dx \quad (2.14)$$

PART II: Suppose first that $\theta < T(x) < x$; then

$$|T(x) - \theta| < |x - \theta|. \quad (2.15)$$

On the other hand, when $T(x) \leq \theta < x$, then

$$0 \leq \mu(x) < T(x) \leq \theta < x$$

in view of (2.6), so that the relation

$$|\mu(x) - \theta| \leq |x - \theta| \quad (2.16)$$

implies (2.15).

Hence, all told,

$$(2.16) \Rightarrow (2.15) \quad (2.17)$$

when $x - \theta > 0$.

But, in view of (2.6),

$$0 \leq \mu(x) < T(x) < x \leq \theta, \quad (2.18)$$

when $x - \theta \leq 0$, so that neither (2.15) nor (2.16) can hold.

Thus (2.17) holds regardless of the sign of $x - \theta$, so that

$$\int_B f(x; \theta) dx < \int_A f(x; \theta) dx, \quad (2.19)$$

where $A = \{x : |T(x) - \theta| < |x - \theta|\}$ and $B = \{x : |\mu(x) - \theta| \leq |x - \theta|\}$. Since $f(x; \theta)$ is supported by the positive real line, it remains to show that B is a proper subset of A , with $A - B$ an open subset of $R^+ = \{x : 0 < x < +\infty\}$.

To that end, consider the point $x_\theta = \min(2\theta, \lambda(\theta))$, for which relation (2.4) implies that

$$0 < x_\theta - \theta = \theta - \mu(x_\theta), \quad (2.20)$$

which, in view of (2.6), implies that

$$x_\theta - \theta > |\theta - T(x_\theta)|. \quad (2.21)$$

Now $x + \mu(x)$ is increasing in view of (2.1) and (2.4), so that (2.20) ensures that there is an x_θ^* , slightly smaller than x_θ , for which

$$0 < x_\theta^* - \theta < \theta - \mu(x_\theta^*) \quad (2.22)$$

and for which the inequality (2.21) is maintained:

$$x_\theta^* - \theta > |\theta - T(x_\theta^*)|. \quad (2.23)$$

Relations (2.22) and (2.23) place x_θ^* in $A - B$, so that $A - B$ is an open subset of R^+ and it follows that

$$\int_{\{x : |\mu(x) - \theta| \leq |x - \theta|\}} f(x; \theta) dx < \int_{\{x : |T(x) - \theta| < |x - \theta|\}} f(x; \theta) dx. \quad (2.24)$$

(Q.E.D.)

Theorem 2.6 *Consider a scalar statistic X , with density $f(x; \theta)$ depending on a scalar parameter θ , $-\infty < \theta < +\infty$, where, for all θ , the support of $f(x; \theta)$ is the*

real line. Suppose that there is a continuous increasing function $\lambda(\theta)$ such that

$$\lambda(\theta) < \theta, \quad \theta < 0 \quad (2.25a)$$

$$\lambda(0) = 0, \quad (2.25b)$$

$$\lambda(\theta) > \theta, \quad \theta > 0 \quad (2.25c)$$

and

$$\int_{\lambda(\theta)}^0 f(x; \theta) dx \leq \frac{1}{2}, \quad \theta \leq 0 \quad (2.26a)$$

$$\int_0^{\lambda(\theta)} f(x; \theta) dx \leq \frac{1}{2}, \quad \theta \geq 0. \quad (2.26b)$$

Define the continuous function $\mu(x)$:

$$\mu(x) = \min(0, \mu_\lambda(x)), \quad x \leq 0,$$

$$\mu(x) = \max(0, \mu_\lambda(x)), \quad x \geq 0,$$

for which, in view of (2.1) and (2.25) with arguments x ,

$$x < \mu(x), \quad x < 0,$$

$$\mu(0) = 0,$$

$$x > \mu(x), \quad x > 0.$$

Then any estimator $T(X)$ of θ , with $T(\cdot)$ continuous and

$$x < T(x) < \mu(x), \quad x < 0 \quad (2.27a)$$

$$T(0) = 0. \quad (2.27b)$$

$$\mu(x) < T(x) < x, \quad x > 0, \quad (2.27c)$$

Pitman-dominates the estimator X ; in other words, for $\theta \in (-\infty, +\infty)$,

$$\Pr_\theta(|T(X) - \theta| < |X - \theta|) > \frac{1}{2}. \quad (2.28)$$

Proof: When $\theta = 0$, in view of (2.27), it is clear that

$$\Pr_{\theta}(|T(X)| < |X|) = \int_{-\infty}^0 f(x; \theta) dx + \int_0^{+\infty} f(x; \theta) dx = 1$$

and thus (2.28) holds.

As to $\theta \neq 0$, the symmetry built into our formulations leads to identical argument for $\theta < 0$ and $\theta > 0$, and we choose the latter.

Analogously to our approach in Lemma 2.1, we see that, in view of (2.26b), it suffices to show that

$$\int_{-\infty}^0 f(x; \theta) dx + \int_{\lambda(\theta)}^{+\infty} f(x; \theta) dx \leq \int_{\{x : |\mu(x) - \theta| \leq |x - \theta|\}} f(x; \theta) dx \quad (2.29)$$

and that

$$\int_{\{x : |\mu(x) - \theta| \leq |x - \theta|\}} f(x; \theta) dx < \int_{\{x : |T(x) - \theta| < |x - \theta|\}} f(x; \theta) dx . \quad (2.30)$$

As in Lemma 2.1, we treat the respective verifications of (2.29) and (2.30) in two parts:

PART I: Relation (2.29) will be verified if

$$x < 0 \quad (2.31)$$

and

$$x \geq \lambda(\theta) \quad (2.32)$$

both imply that

$$|\mu(x) - \theta| \leq |x - \theta| . \quad (2.33)$$

That (2.31) implies (2.33) follows immediately from (2.27a), while the demonstration that (2.32) implies (2.33) is precisely that of PART I of Lemma 2.1.

PART II: The open set analogous to the open set $A - B$ of Lemma 2.1 is seen, by the same argument as before, to contain a point x_θ^* . (Q.E.D.)

Note 2.1: $T(\cdot)$ in effect “shrinks” a statistic toward zero. In keeping with the fact that, generally, shrinkage need not always be toward the origin in location problems, the construction may, by a translate, be focused on statistics $T(\cdot)$ satisfying $T(c) = c$, c arbitrary, rather than $T(0) = 0$.

Theorem 2.6 is now illustrated by two examples. For each of these, the essential step is to identify a continuous increasing function $\lambda(\theta)$, $-\infty < \theta < +\infty$, satisfying (2.25) and (2.26).

Example 2.3 This is the case treated in David and Salem [5] with a symmetry restriction on $f(x; \theta)$ and monotonicity restriction on $T(\cdot)$. Let X be an observation from a density $f(x; \theta)$ supported by the real line, and θ be the median. Since

$$\int_{-\infty}^{\theta} f(x; \theta) dx = \frac{1}{2},$$

it is clear that

$$\int_{\theta}^0 f(x; \theta) dx < \frac{1}{2}, \quad \theta \leq 0 \quad (2.34a)$$

and

$$\int_0^{\theta} f(x; \theta) dx < \frac{1}{2}, \quad \theta \geq 0. \quad (2.34b)$$

Without loss of generality, consider (2.34b), which guarantees that there is a contin-

uous function $\lambda^*(\theta)$, $\lambda^*(\theta) > \theta \geq 0$, such that

$$\int_0^{\lambda^*(\theta)} f(x; \theta) dx = \frac{1}{2} , \quad (2.35)$$

and also a continuous increasing function $\lambda(\theta)$ satisfying (2.25b) and (2.25c) :

$$\lambda(0) = 0 , \quad (2.36a)$$

$$\theta < \lambda(\theta) \leq \lambda^*(\theta) , \theta > 0 , \quad (2.36b)$$

as well as (2.26b) in view of (2.35) and (2.36). Then, in view of Theorem 2.6, any continuous function $T(X)$ satisfying (2.27) is Pitman-closer to θ than is X . \square

In particular, a class of estimators of θ Pitman-dominating the sample mean of a normal density with known variance can be constructed by this example. Let X be an observation from a normal density $\phi(x; \theta)$. Figure 2.1 shows the continuous function $\lambda^*(\cdot)$ satisfying (2.35) and its negative analogue. Figure 2.2 shows $\mu(x)$ (we call this “boundary function”). Table 2.1 shows the value of the median-unbiased estimator X and the value of the boundary function $\mu(x)$.

As noted in Section 1.2.3, Salem [30] and Efron [7] constructed a shrinkage estimator dominating an observation X from a normal population with unknown mean θ and unit variance. Figure 2.3 shows the comparisons of Salem and Efron estimators, and David’s [5] boundary function for the normal density with unit variance, and indicates that Salem and Efron estimators are included in the class of shrinkage estimators constructed by applying Theorem 2.6. Figure 2.4 shows the comparisons of $P_0(\theta) = 1 - \Pr_\theta(0 < X < \theta)$, $P_1(\theta) = \Pr_\theta(|E(X) - \theta| < |X - \theta|)$, $P_2(\theta) = \Pr_\theta(|S(X) - \theta| < |X - \theta|)$, and $P_3(\theta) = \Pr_\theta(|\mu(X) - \theta| < |X - \theta|)$, where

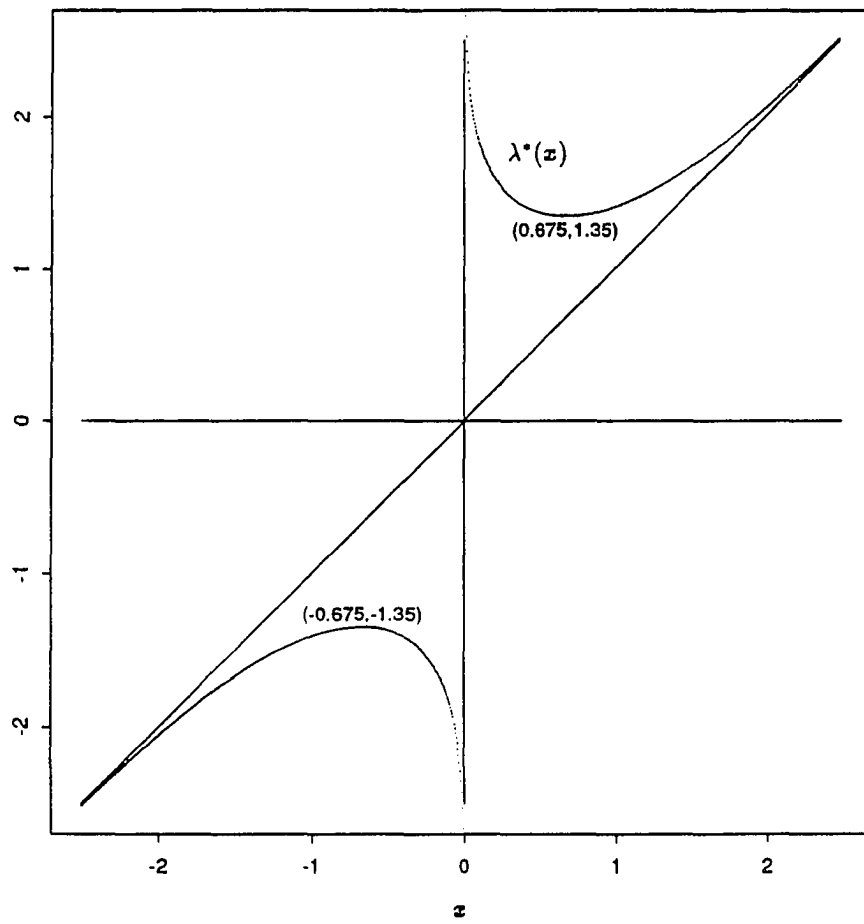


Figure 2.1: The graph of $\lambda^*(x)$ for the normal density with unit variance

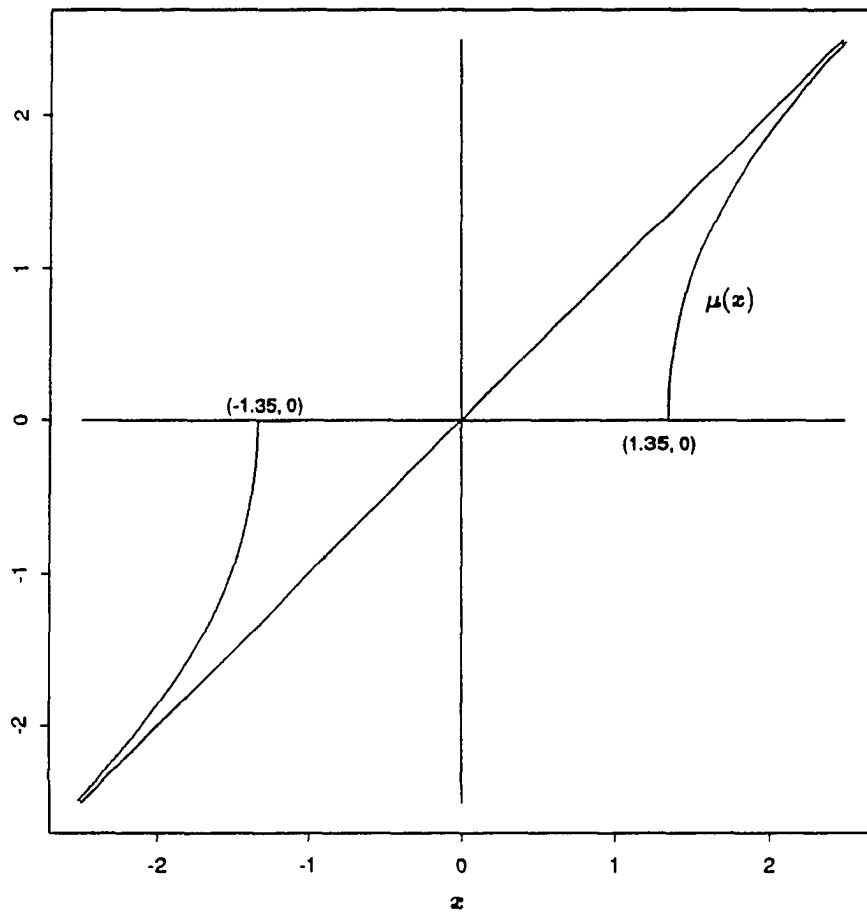


Figure 2.2: The graph of the boundary function $\mu(x)$

Table 2.1: The boundary function $\mu(x)$ for the normal density with unit variance

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.3	0.0	0.0	0.0	0.0	0.0	0.185	0.305	0.386	0.456	0.516
1.4	0.566	0.616	0.665	0.705	0.745	0.777	0.816	0.847	0.877	0.907
1.5	0.937	0.967	0.997	1.026	1.048	1.076	1.105	1.126	1.147	1.175
1.6	1.196	1.217	1.237	1.258	1.278	1.305	1.325	1.345	1.358	1.378
1.7	1.398	1.417	1.436	1.456	1.469	1.488	1.507	1.526	1.538	1.557
1.8	1.576	1.588	1.607	1.619	1.637	1.656	1.668	1.686	1.698	1.716
1.9	1.728	1.746	1.757	1.775	1.787	1.799	1.816	1.828	1.845	1.857
2.0	1.869	1.886	1.897	1.909	1.926	1.937	1.949	1.966	1.977	1.988
2.1	1.999	2.016	2.027	2.039	2.055	2.066	2.077	2.088	2.099	2.116
2.2	2.127	2.138	2.149	2.165	2.176	2.187	2.198	2.209	2.219	2.236
2.3	2.246	2.257	2.268	2.278	2.289	2.305	2.316	2.326	2.337	2.348
2.4	2.358	2.369	2.379	2.395	2.406	2.416	2.427	2.437	2.448	2.458
2.5	2.469	2.479	2.490	2.505	2.516	2.526	2.536	2.547	2.557	2.568
2.6	2.578	2.588	2.599	2.609	2.619	2.630	2.645	2.655	2.666	2.676
2.7	2.686	2.696	2.707	2.717	2.727	2.737	2.748	2.758	2.768	2.778
2.8	2.788	2.799	2.809	2.819	2.829	2.839	2.850	2.860	2.875	2.885
2.9	2.890	2.905	2.916	2.926	2.936	2.946	2.956	2.966	2.986	2.987

$E(X)$ and $S(X)$ are shrinkage estimators of θ constructed by Efron and Salem, respectively. It is noted that $P_4(\theta) = \Pr_\theta(|S(X) - \theta| < |E(X) - \theta|)$ is greater than 0.5. Figure 2.5 shows the graph of $P_4(\theta)$.

Example 2.4 Consider the “non-central t ” case of estimating the non-centrality parameter $\delta \equiv 2^{1/2}\theta/\sigma$ using $2^{1/2}\bar{X}/S$ for sample size two, where the X_i ’s form a random sample from a normal distribution with mean θ and standard deviation σ , and S^2 is the unbiased estimator of σ^2 . Note that $2^{1/2}\bar{X}/S$ is distributed as non-central t with 1 degree of freedom and non-centrality parameter δ . Without loss of generality suppose that δ is positive. Conditioning on the random variable S^2/σ^2 which follows the chi-square distribution with 1 degree of freedom, one finds that

$$\begin{aligned} & \Pr_\delta(0 < 2^{1/2}\bar{X}/S < \delta) \\ &= \Pr_\delta\left\{-\delta < \frac{2^{1/2}(\bar{X} - \theta)}{\sigma} < \delta\left(\frac{S}{\sigma} - 1\right)\right\} \\ &= \int_{-1}^{\infty} \Phi(\delta y)f(y)dy - \Phi(-\delta) \quad , \end{aligned} \tag{2.37}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function and $f(y) = \left(\frac{2}{\pi}\right)^{1/2} \exp\left(-\frac{1}{2}(y+1)^2\right)$ is the probability density function of $Y = X^{1/2} - 1$, where the distribution of X is chi-square with 1 degree of freedom. $\Phi(\delta y)$ is convex and increasing on $-1 < y < 0$; thus it is bounded above by $(\Phi(0) - \Phi(-\delta))y + \Phi(-\delta)$ on $-1 < y < 0$. Hence,

$$\begin{aligned} & \int_{-1}^{\infty} \Phi(\delta y)f(y)dy \\ &< \int_{-1}^0 \left\{\left(\frac{1}{2} - \Phi(-\delta)\right)y + \frac{1}{2}\right\} f(y)dy + \int_0^{\infty} f(y)dy \\ &= \left\{\frac{1}{2} - \Phi(-\delta)\right\} \int_{-1}^0 yf(y)dy + \frac{1}{2} + \frac{1}{2} \int_0^{\infty} f(y)dy \end{aligned}$$

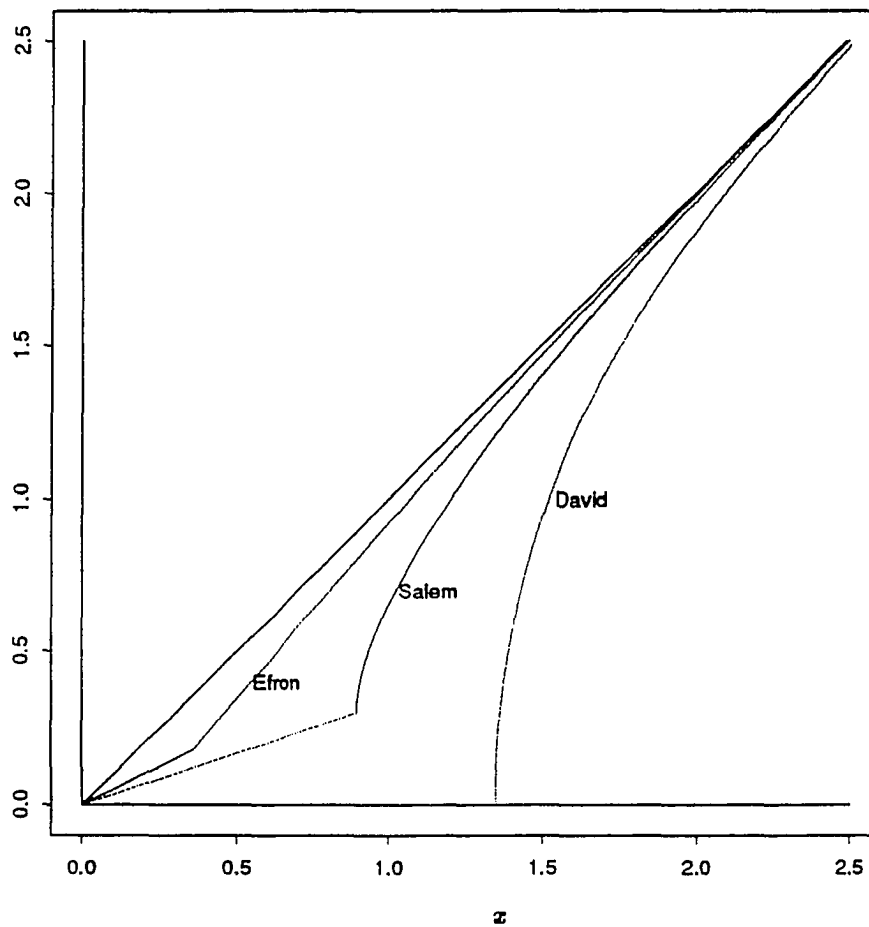


Figure 2.3: Comparison of Salem and Efron estimators, and David's boundary function, on the positive real line for the normal density with unit variance

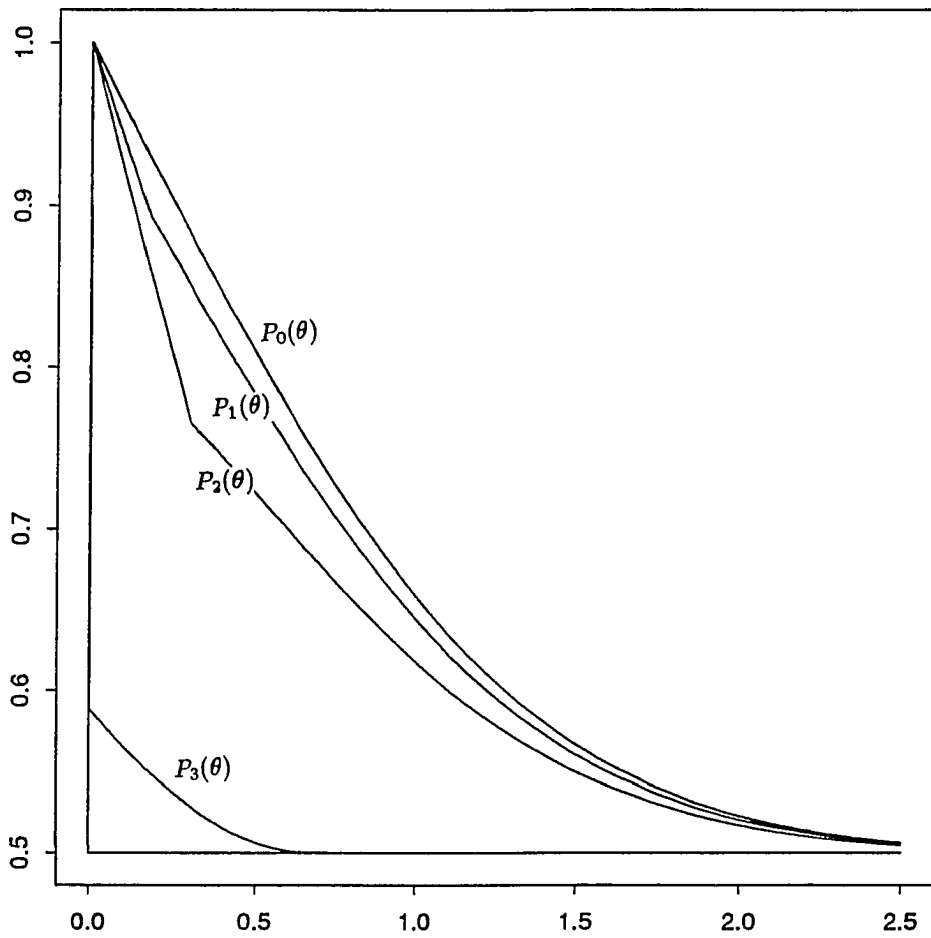


Figure 2.4: Comparison of $P_0(\theta)$, $P_1(\theta)$, $P_2(\theta)$, and $P_3(\theta)$, on the positive real line for the normal density with unit variance

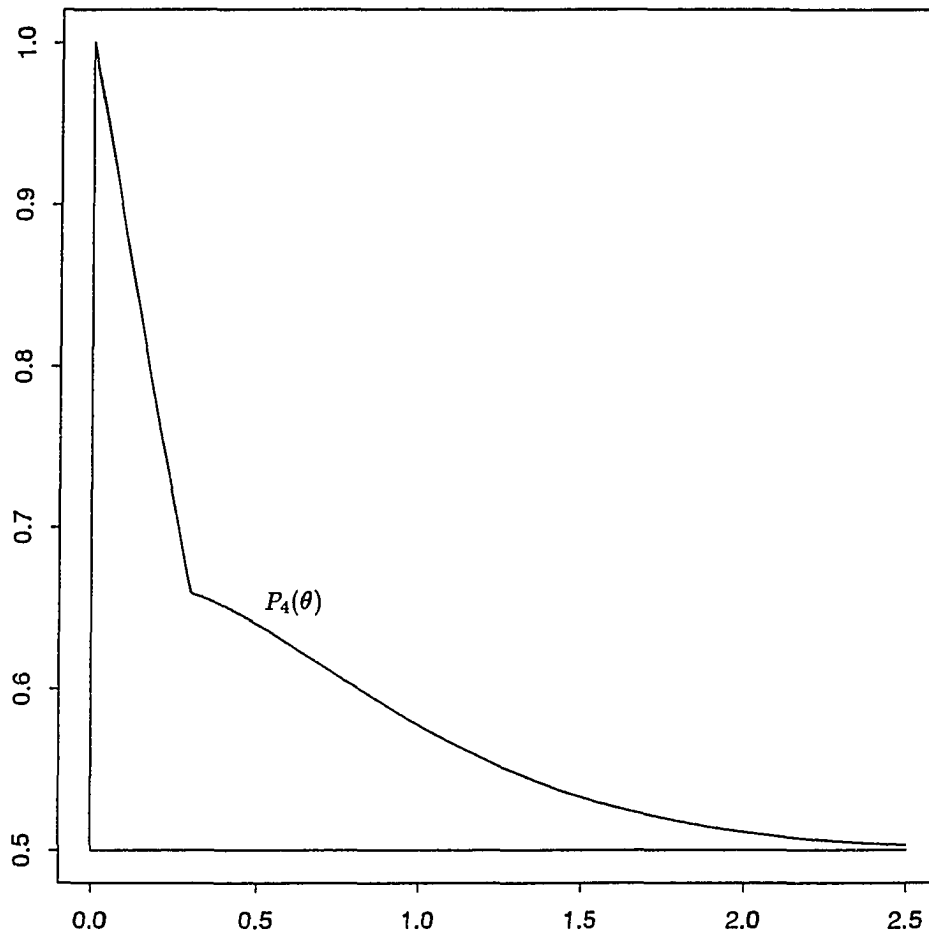


Figure 2.5: The graph of $P_4(\theta)$, on the positive real line for the normal density with unit variance

$$\begin{aligned}
&= \{1 - 2\Phi(-\delta)\} \{\phi(0) - \phi(1)\} + \frac{1}{2} + (1 - \Phi(1)) \\
&= \{\phi(0) - \phi(1) + 2(1 - \Phi(1))\} - 2 \left\{ \phi(0) - \phi(1) - \left(\Phi(1) - \frac{1}{2} \right) \right\} \Phi(-\delta) \\
&< \frac{1}{2} + \Phi(-\delta), \tag{2.38}
\end{aligned}$$

where $\phi(\cdot)$ is the standard normal probability density function. Thus, in view of (2.37) and (2.38),

$$\Pr_{\delta}(0 < 2^{1/2} \bar{X}/S < \delta) < \frac{1}{2}, \tag{2.39}$$

and also

$$\Pr_{\delta}(0 < 2^{1/2} \bar{X}/S < +\infty) = 1 - \Phi(-\delta) \geq \frac{1}{2}. \tag{2.40}$$

Hence, in view of (2.39) and (2.40), one can define a continuous increasing function $\lambda(\delta)$ satisfying (2.25), and (2.26) with equality, with argument δ :

$$\Pr_{\delta}(0 < 2^{1/2} \bar{X}/S \leq \lambda(\delta)) = \frac{1}{2}.$$

Therefore, in view of Theorem 2.6, any continuous function $T(2^{1/2} \bar{X}/S)$, with $T(\cdot)$ satisfying (2.27), is Pitman-closer than $2^{1/2} \bar{X}/S$ to δ . \square

Note 2.2: The case of $n = 3$ is carried out similarly and the case of $n = \infty$ is formally covered by Example 2.3; also, numerical investigations indicate that the construction may be carried out for any sample size.

Theorem 2.7 implements Note 2.1, and, in addition, introduces half-infinite domains; its proof essentially follows that of Theorem 2.6, and is not given.

Theorem 2.7 *Consider a scalar statistic X , with density $f(x; \theta)$ depending on a scalar parameter θ , $a < \theta < +\infty$, where a is the lower limit of the domain of θ ,*

and the support of $f(x; \theta)$ is $(a, +\infty)$. Suppose that, for some constant $c > a$, there is a continuous increasing function $\lambda(\theta), \theta > a$, such that

$$\lambda(\theta) < \theta, \quad a < \theta < c \quad (2.41a)$$

$$\lambda(c) = c, \quad (2.41b)$$

$$\lambda(\theta) > \theta, \quad \theta > c \quad (2.41c)$$

and

$$\int_{\lambda(\theta)}^c f(x; \theta) dx \leq \frac{1}{2}, \quad a < \theta \leq c \quad (2.42a)$$

$$\int_c^{\lambda(\theta)} f(x; \theta) dx \leq \frac{1}{2}, \quad \theta \geq c. \quad (2.42b)$$

Define the continuous function $\mu(x)$:

$$\mu(x) = \min(c, \mu_\lambda(x)), \quad a < x \leq c,$$

$$\mu(x) = \max(c, \mu_\lambda(x)), \quad x \geq c,$$

for which, in view of (2.1) and (2.41) with arguments x ,

$$x < \mu(x), \quad a < x < c,$$

$$\mu(c) = c,$$

$$x > \mu(x), \quad x > c.$$

Then any estimator $T(X)$ of θ , with $T(\cdot)$ continuous and

$$x < T(x) < \mu(x), \quad a < x < c \quad (2.43a)$$

$$T(c) = c, \quad (2.43b)$$

$$\mu(x) < T(x) < x, \quad x > c, \quad (2.43c)$$

Pitman-dominates the estimator X ; in other words, for $\theta > a$,

$$\Pr_{\theta}(|T(X) - \theta| < |X - \theta|) > \frac{1}{2}. \quad (2.44)$$

Theorem 2.7 is now illustrated by an example. Analogous to the two previous examples, the essential step is to identify a continuous increasing function $\lambda(\theta)$, $\theta > a$, satisfying (2.41) and (2.42).

Example 2.5 Let X be a median-unbiased for the unknown positive parameter θ of a density $f(x; \theta)$ supported by the positive real line, so that $a = 0$. Since

$$\int_0^{\theta} f(x; \theta) dx = \frac{1}{2},$$

it is clear that, for any $c > 0$,

$$\int_{\theta}^c f(x; \theta) dx < \frac{1}{2}, \quad 0 < \theta \leq c \quad (2.45a)$$

and

$$\int_c^{\theta} f(x; \theta) dx < \frac{1}{2}, \quad c \leq \theta < \infty. \quad (2.45b)$$

Without loss of generality, consider (2.45b), which guarantees that there is a continuous function $\lambda^*(\theta)$, $\lambda^*(\theta) > \theta \geq c$, such that

$$\int_c^{\lambda^*(\theta)} f(x; \theta) dx = \frac{1}{2}, \quad (2.46)$$

and also a continuous increasing function $\lambda(\theta)$ satisfying (2.41b) and (2.41c):

$$\lambda(c) = c, \quad (2.47a)$$

$$\theta < \lambda(\theta) \leq \lambda^*(\theta), \quad \theta > c, \quad (2.47b)$$

as well as (2.42b) in view of (2.46) and (2.47). Then, in view of Theorem 2.7, any continuous function $T(X)$ satisfying (2.43) is Pitman-closer to θ than is X . \square

In particular, Example 2.5 pertains to constructing a class of estimators Pitman-dominating the minimal sufficient median-unbiased estimator of the variance σ^2 of a normal distribution with known mean. Example 2.5 also pertains to constructing a class of estimators Pitman-dominating the multiple of the ratio of two independent sample variances that is median-unbiased for the corresponding population variance ratio.

Let S^2 be the sufficient unbiased estimator of σ^2 of a normal distribution with unknown mean (i.e., $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2 / (n - 1)$). Then, in view of Proposition 2.1,

$$\Pr_{\sigma}(0 < S^2 < \sigma^2) = \Pr_{\sigma}(0 < \frac{(n-1)S^2}{\sigma^2} < n-1) > \frac{1}{2} .$$

Thus, there exist $k_n, 0 < k_n < 1$, such that

$$\Pr_{\sigma}(0 < \frac{S^2}{k_n} < \sigma^2) = \frac{1}{2} .$$

Therefore, $\frac{S^2}{k_n}$ is a median-unbiased estimator of σ^2 . Now it is clear that, for any $c > 0$, there exist a continuous function $\lambda^*(\sigma^2)$ such that

$$\Pr_{\sigma}(c < \frac{S^2}{k_n} < \lambda^*(\sigma^2)) = \frac{1}{2} , \lambda^*(\sigma^2) > \sigma^2 > c ,$$

$$\lambda^*(c) = c ,$$

$$\Pr_{\sigma}(\lambda^*(\sigma^2) < \frac{S^2}{k_n} < c) = \frac{1}{2} , \lambda^*(\sigma^2) < \sigma^2 < c .$$

Figure 2.6 shows the graph of $\lambda^*(x)$ with $df = 10$ and $c = 1$. Figure 2.7 shows the boundary function $\mu(x)$ with $df = 10$ and $c = 1$.

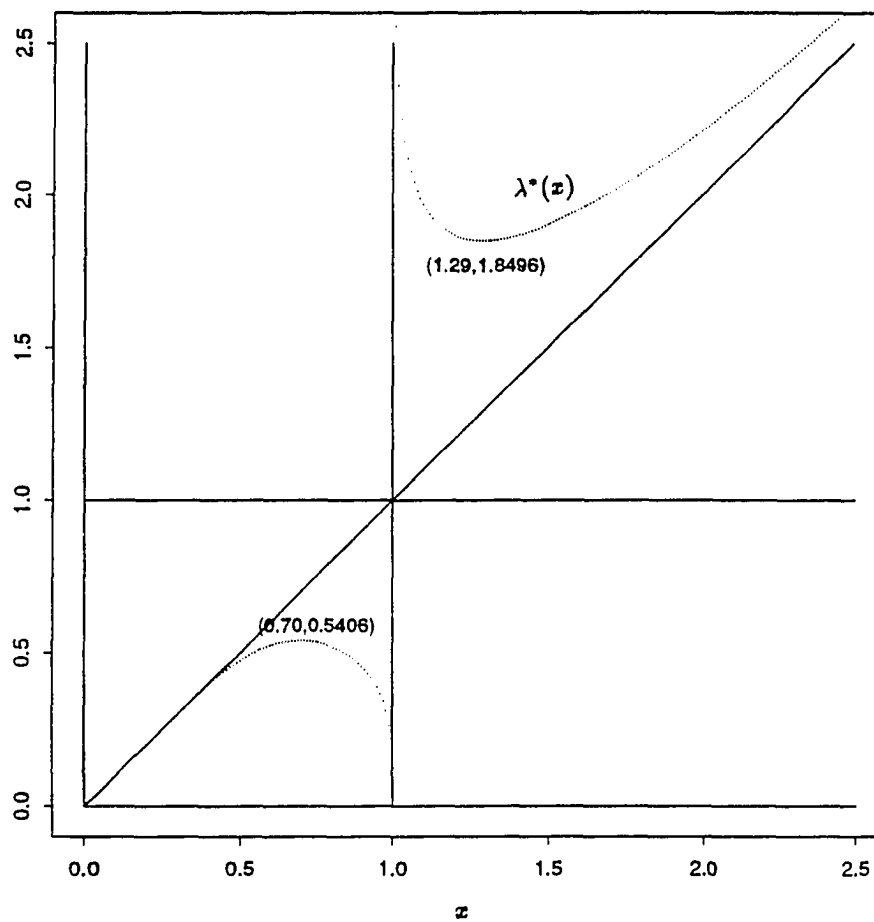


Figure 2.6: $\lambda^*(x)$ for the chi-square density with $df = 10$ and $c = 1$

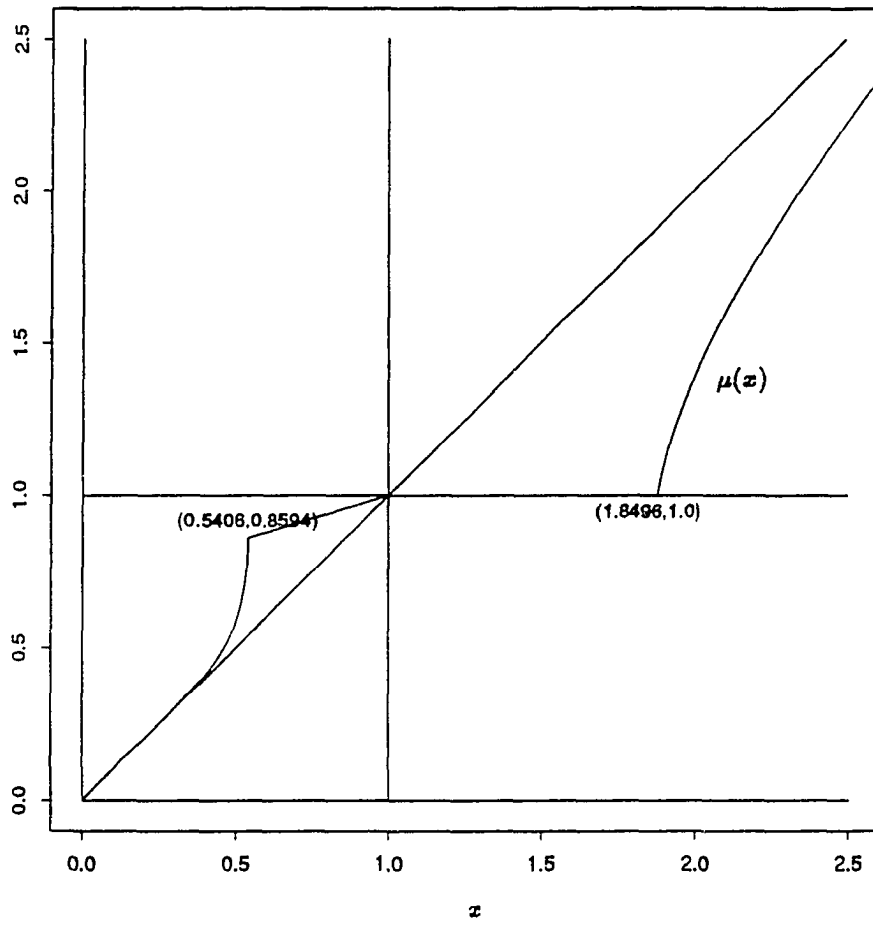


Figure 2.7: $\mu(x)$ for the chi-square density with $df = 10$ and $c = 1$

2.3.3 Shrinkage Constructions for Generalized Pitman Domination

In this section, we consider a loss function of the form $L(x, \theta) = h(x - \theta)$ where $h(y) = r(y)$ on $[0, +\infty)$ and $s(-y)$ on $(-\infty, 0]$, with $r(y)$ and $s(y)$ continuous increasing on $[0, +\infty)$ and $r(0) = s(0) = 0$.

For Theorem 2.8, the following definitions are needed: Let a_1 and a_2 satisfy $-\infty \leq a_1 < a_2 \leq +\infty$, and let I be the interval (a_1, a_2) . Let X , with density $f(x; \theta_1)$ supported by I , have median θ for θ restricted to I . Further, for given $c \in I$, let $\lambda(\cdot)$ be any continuous increasing function satisfying

$$\int_{\lambda(\theta)}^c f(x; \theta) dx \leq 1/2, \quad a_1 < \theta \leq c, \quad (2.48a)$$

$$\int_c^{\lambda(\theta)} f(x; \theta) dx \leq 1/2, \quad c \leq \theta < a_2, \quad (2.48b)$$

$$\lambda(\theta) < \theta, \quad a_1 < \theta < c, \quad (2.48c)$$

$$\lambda(c) = c, \quad (2.48d)$$

$$\lambda(\theta) > \theta, \quad c < \theta < a_2; \quad (2.48e)$$

and let $\mu(x)$ on I be $\lambda^{-1}(x)$.

Theorem 2.8 *Let X , median-unbiased for θ , $\theta \in I$, have density $f(x; \theta)$ with support I for all θ . Then, for $c \in (a_1, a_2)$, any estimator $T(X)$ of θ with $T(\cdot)$ continuous and*

$$x < T(x) < \mu(x), \quad a_1 < x < c \quad (2.49a)$$

$$T(c) = c, \quad (2.49b)$$

$$\mu(x) < T(x) < x, \quad c < x < a_2, \quad (2.49c)$$

Pitman-dominates the estimator X with respect to any loss function of type L ; in other words, for $\theta \in (a_1, a_2)$,

$$\Pr_{\theta}(L(T(X), \theta) < L(X, \theta)) > 1/2. \quad (2.50)$$

Note that $T(\cdot)$ satisfying (2.49) always exists in view of the definition of $\lambda(\cdot)$.

Proof: When $\theta = c$, in view of (2.49) and the properties of the loss function $L(\cdot, \cdot)$, it is clear that

$$\Pr_{\theta}(L(T(X), \theta) < L(X, \theta)) = \int_{a_1}^c f(x; \theta) dx + \int_c^{a_2} f(x; \theta) dx = 1$$

and thus (2.50) holds.

As to $\theta \neq c$, the argument for $a_1 < \theta < c$ is essentially identical to that for $c < \theta < a_2$, and we choose the latter.

For $\theta \in (c, a_2)$, in view of (2.48b) and (2.50), it suffices to show that, given any $L(\cdot, \cdot)$,

$$\int_{a_1}^c f(x; \theta) dx + \int_{\lambda(\theta)}^{a_2} f(x; \theta) dx \leq \int_B f(x; \theta) dx \quad (2.51)$$

and

$$\int_B f(x; \theta) dx < \int_A f(x; \theta) dx \quad (2.52)$$

, where $A = \{x : L(T(x), \theta) < L(x, \theta)\}$ and $B = \{x : L(\mu(x), \theta) \leq L(x, \theta)\}$.

With regard to (2.51), it is clear, in view of (2.48c), (2.48d), (2.48e), and (2.49), that both

$$a_1 < x < c$$

and

$$\lambda(\theta) \leq x < a_2$$

imply

$$L(\mu(x), \theta) \leq L(x, \theta) .$$

Hence, (2.51) holds.

With regard to (2.52), it is not difficult to see that B is a proper subset of A , with $A - B$ an open subset of I , which implies (2.52). **(Q.E.D)**

Note 2.3: In the symmetric case $r(\cdot) = s(\cdot)$, Theorem 2.8 still holds with $\mu(x)$ defined as

$$\mu(x) = \min(c, 2\lambda^{-1}(x) - x) , \quad a_1 < x \leq c ,$$

$$\mu(x) = \max(c, 2\lambda^{-1}(x) - x) , \quad c \leq x < a_2 ,$$

furnishing a larger class of estimators $T(X)$ Pitman-dominating X .

CHAPTER 3. UNIVARIATE SHRINKAGE CONSTRUCTIONS FOR LOCATION-SCALE FAMILIES

3.1 Introduction

Let X be an estimator of the location parameter θ taken from a location-scale family of density functions $f(x; \theta, \sigma)$, where both θ and σ are unknown. Since the density function of X depends on both θ and σ , we cannot apply the results in the previous chapter.

In section 3.2, we essentially show that a shrinkage estimator $T(X)$ of θ Pitman-dominating X for any fixed σ_0 also dominates X in the Pitman sense for any $\sigma > \sigma_0$, so that, for such a $T(X)$,

$$\Pr_{(\theta, \sigma)}(|T(X) - \theta| < |X - \theta|) > \frac{1}{2} ,$$

for all θ and all $\sigma \geq \sigma_0$.

On the other hand, in section 3.3, we show that it is unlikely that any $T(X)$ can be found, for which the restriction $\sigma \geq \sigma_0$ can be removed. In particular, it is shown that, within a certain class of functions $T(X)$, there is always a value θ_0 of θ such that

$$\lim_{\sigma \rightarrow 0} \Pr_{(\theta_0, \sigma)}(|X - \theta| < |T(X) - \theta|) = 1 .$$

3.2 Shrinkage Constructions for the Case of Bounded Below Scale

Parameter

Let X be a median-unbiased estimator of the location parameter θ taken from a location-scale family of density functions $f(x; \theta, \sigma)$, where both θ and σ are unknown. If there exists a lower bound of σ other than zero then we can construct shrinkage estimators of θ Pitman-dominating X in the following way.

Let σ_0 be the lower bound of σ , i.e., $\sigma \geq \sigma_0$. It is clear that

$$\Pr_{(\theta, \sigma)}(0 < X < \theta) < \frac{1}{2}, \quad 0 < \theta < +\infty, \quad 0 < \sigma < +\infty, \quad (3.1a)$$

$$\Pr_{(\theta, \sigma)}(\theta < X < 0) < \frac{1}{2}, \quad -\infty < \theta < 0, \quad 0 < \sigma < +\infty. \quad (3.1b)$$

For the following lemmas, we define $\lambda_{\sigma}^*(\cdot)$ to be a continuous function satisfying, for given σ , $\sigma \in [\sigma_0, +\infty)$,

$$\int_{\lambda_{\sigma}^*(\theta)}^0 f(x; \theta, \sigma) dx = 1/2, \quad -\infty < \theta \leq 0, \quad (3.2a)$$

$$\int_0^{\lambda_{\sigma}^*(\theta)} f(x; \theta, \sigma) dx = 1/2, \quad 0 \leq \theta < +\infty, \quad (3.2b)$$

$$\lambda_{\sigma}^*(\theta) < \theta, \quad -\infty < \theta < 0, \quad (3.2c)$$

$$\lambda_{\sigma}^*(0) = 0, \quad (3.2d)$$

$$\lambda_{\sigma}^*(\theta) > \theta, \quad 0 < \theta < +\infty. \quad (3.2e)$$

Lemma 3.1 For given $\sigma_1 > \sigma_0$, $\lambda_{\sigma_1}^*(\theta) > \lambda_{\sigma_0}^*(\theta)$ if $\theta > 0$, and $\lambda_{\sigma_1}^*(\theta) < \lambda_{\sigma_0}^*(\theta)$ if $\theta < 0$.

Proof: In the case that $\theta > 0$. In view of (3.2b), it is true that, for given σ , $\sigma \in [\sigma_0, +\infty)$,

$$\begin{aligned}
& \int_0^{\lambda_{\sigma}^*(\theta)} f(x; \theta, \sigma) dx \\
&= \Pr_{(\theta, \sigma)}(0 < X < \lambda_{\sigma}^*(\theta)) \\
&= \Pr_{(\theta, \sigma)}\left(-\frac{\theta}{\sigma} < \frac{X - \theta}{\sigma} < \frac{\lambda_{\sigma}^*(\theta) - \theta}{\sigma}\right) \\
&= \Phi\left(\frac{\lambda_{\sigma}^*(\theta) - \theta}{\sigma}\right) - \Phi\left(-\frac{\theta}{\sigma}\right) = \frac{1}{2}, \tag{3.3}
\end{aligned}$$

where $\Phi(\cdot)$ is the cumulative distribution of standardized random variable $\frac{X - \theta}{\sigma}$.

Now, in view of (3.3), it is true that

$$\Phi\left(-\frac{\theta}{\sigma_1}\right) > \Phi\left(-\frac{\theta}{\sigma_0}\right)$$

and thus

$$\Phi\left(\frac{\lambda_{\sigma_1}^*(\theta) - \theta}{\sigma_1}\right) > \Phi\left(\frac{\lambda_{\sigma_0}^*(\theta) - \theta}{\sigma_0}\right),$$

so that,

$$\frac{\lambda_{\sigma_1}^*(\theta) - \theta}{\sigma_1} > \frac{\lambda_{\sigma_0}^*(\theta) - \theta}{\sigma_0}. \tag{3.4}$$

In view of (3.2e) and (3.4), we have that

$$\begin{aligned}
\lambda_{\sigma_1}^*(\theta) &> \left(\frac{\sigma_1}{\sigma_0}\right)(\lambda_{\sigma_0}^*(\theta) - \theta) + \theta \\
&= \lambda_{\sigma_0}^*(\theta) + \left(\frac{\sigma_1}{\sigma_0} - 1\right)(\lambda_{\sigma_0}^*(\theta) - \theta) \\
&> \lambda_{\sigma_0}^*(\theta).
\end{aligned}$$

In the case that $\theta < 0$, similar argument holds. (Q.E.D)

Now, for given σ , $\sigma \in [\sigma_0, +\infty)$, let $\lambda_\sigma(\cdot)$ be as defined with respect to $\lambda_\sigma^*(\cdot)$ as in section 2.3.2:

$$\lambda_\sigma^*(\theta) \leq \lambda_\sigma(\theta) < \theta , \quad -\infty < \theta < 0 , \quad (3.5a)$$

$$\lambda_\sigma(0) = 0 , \quad (3.5b)$$

$$\theta < \lambda_\sigma(\theta) \leq \lambda_\sigma^*(\theta) , \quad 0 < \theta < +\infty . \quad (3.5c)$$

Then, in view of Lemma 3.1 and (3.5), for given $\sigma_1 > \sigma_0$, there exist $\lambda_{\sigma_0}(\theta)$ and $\lambda_{\sigma_1}(\theta)$ satisfying

$$\lambda_{\sigma_1}(\theta) < \lambda_{\sigma_0}(\theta) < \theta , \quad -\infty < \theta < 0 , \quad (3.6a)$$

$$\lambda_{\sigma_0}(0) = \lambda_{\sigma_1}(0) = 0 , \quad (3.6b)$$

$$\theta < \lambda_{\sigma_0}(\theta) < \lambda_{\sigma_1}(\theta) , \quad 0 < \theta < +\infty . \quad (3.6c)$$

Now, for given σ , define \mathcal{T}_σ to be the class of estimators $T(X)$ identified in Theorem 2.6, satisfying

$$x < T(x) < \lambda_\sigma^{-1}(x) , \quad -\infty < x < 0 \quad (3.7a)$$

$$T(0) = 0 , \quad (3.7b)$$

$$\lambda_\sigma^{-1}(x) < T(x) < x , \quad 0 < x < +\infty , \quad (3.7c)$$

where $\lambda_\sigma^{-1}(x)$ is the inverse function of $\lambda_\sigma(x)$ with respect to x for given σ . Then we have the following lemma.

Lemma 3.2 *For given $\sigma_1 > \sigma_0$, $\mathcal{T}_{\sigma_1} \supset \mathcal{T}_{\sigma_0}$.*

Proof: In view of (3.6) and the definition of λ_σ , we have, for given $\sigma_1 > \sigma_0$, that

$$\theta < \lambda_{\sigma_0}^{-1}(\theta) < \lambda_{\sigma_1}^{-1}(\theta) < 0, \quad -\infty < \theta < 0, \quad (3.8a)$$

$$\lambda_{\sigma_0}^{-1}(0) = \lambda_{\sigma_1}^{-1}(0) = 0, \quad (3.8b)$$

$$0 < \lambda_{\sigma_1}^{-1}(\theta) < \lambda_{\sigma_0}^{-1}(\theta) < \theta, \quad 0 < \theta < +\infty. \quad (3.8c)$$

Thus, in view of (3.7) and (3.8), we have that

$$x < T(x) < \lambda_{\sigma_0}^{-1}(x) < \lambda_{\sigma_1}^{-1}(x) < 0, \quad -\infty < x < 0$$

$$T(0) = 0,$$

$$0 < \lambda_{\sigma_1}^{-1}(x) < \lambda_{\sigma_0}^{-1}(x) < T(x) < x, \quad 0 < x < +\infty.$$

Hence, for given $\sigma_1 > \sigma_0$, $\mathcal{T}_{\sigma_1} \supset \mathcal{T}_{\sigma_0}$. (Q.E.D)

Theorem 3.1 *Let X , median-unbiased for a location parameter θ , be taken from a location-scale family of density functions $f(x; \theta, \sigma)$, where both θ and σ are unknown. Then, for all $\sigma_1 \geq \sigma_0$ and any estimator $T_{\sigma_0}(X)$ of θ with $T_{\sigma_0}(X) \in \mathcal{T}_{\sigma_0}$, we have*

$$\Pr_{(\theta, \sigma_1)}(|T_{\sigma_0}(X) - \theta| < |X - \theta|) > 1/2, \quad \forall \theta \in R. \quad (3.9)$$

Proof: In view of Theorem 2.6 in Chapter 2, we have, for given σ_1 , that

$$\Pr_{(\theta, \sigma_1)}(|T(X) - \theta| < |X - \theta|) > 1/2, \quad \forall \theta \in R,$$

for which $T(X) \in \mathcal{T}_{\sigma_1}$. Since, in view of Lemma 3.2, $\mathcal{T}_{\sigma_1} \supset \mathcal{T}_{\sigma_0}$, $T_{\sigma_0}(X) \in \mathcal{T}_{\sigma_1}$.

(Q.E.D)

3.3 Non-existence of Pitman-dominating Estimators

We can construct univariate shrinkage estimators Pitman-dominating an estimator X when the density function of X is from only either location or scale family. However, if the density function of X is from a location-scale family then, under certain conditions, there is no estimator Pitman-dominating X . We study the non-existence of such an estimator that Pitman-dominates X in the following theorem.

Theorem 3.2 *Let X be an observation from the location-scale family of density functions $f(x; \theta, \sigma)$, where both θ and σ are unknown. Let X be absolutely continuous and $T(X)$ arbitrary such that there is an θ_0 with*

$$a \equiv \lim_{x \rightarrow \theta_0 -} T(x) \neq \theta_0 \quad (3.10)$$

and

$$b \equiv \lim_{x \rightarrow \theta_0 +} T(x) \neq \theta_0 . \quad (3.11)$$

Then

$$\lim_{\sigma \rightarrow 0} \Pr_{(\theta_0, \sigma)} (|X - \theta| < |T(X) - \theta|) = 1 .$$

Proof: Let $\eta = \min(|a - \theta_0|, |b - \theta_0|)$. In view of (3.10), for given $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that

$$|T(x) - a| < \epsilon_1$$

whenever

$$\theta_0 - \delta_1 < x < \theta_0 .$$

In view of (3.11), for given $\epsilon_2 > 0$, there exists $\delta_2 > 0$ such that

$$|T(x) - b| < \epsilon_2$$

whenever

$$\theta_0 < x < \theta_0 + \delta_2 \ .$$

Now note that

$$\begin{aligned} T(x) - \theta_0 &= (a - \theta_0) + (T(x) - a) \\ &= (b - \theta_0) + (T(x) - b) \ . \end{aligned}$$

Then

$$|T(x) - \theta_0| \geq |a - \theta_0| - |T(x) - a| \tag{3.12}$$

and

$$|T(x) - \theta_0| \geq |b - \theta_0| - |T(x) - b| \ . \tag{3.13}$$

Recall that $|a - \theta_0| \geq \eta$ and $|b - \theta_0| \geq \eta$. Now in view of (3.12), given $\epsilon_1 = \eta/3$, there exists $\delta_1 > 0$ such that

$$|T(x) - \theta_0| > \frac{2\eta}{3}$$

whenever

$$\theta_0 - \delta_1 < x < \theta_0 \ .$$

Also, in view of (3.13), given $\epsilon_2 = \eta/3$, there exists $\delta_2 > 0$ such that

$$|T(x) - \theta_0| > \frac{2\eta}{3}$$

whenever

$$\theta_0 < x < \theta_0 + \delta_2 \quad .$$

Thus, if

$$\max \left(\theta_0 - \delta_1, \theta_0 - \frac{\eta}{3} \right) < x < \theta_0$$

or

$$\theta_0 < x < \min \left(\theta_0 + \delta_2, \theta_0 + \frac{\eta}{3} \right)$$

then

$$|x - \theta_0| < |T(x) - \theta_0| \quad .$$

Hence we have that

$$\begin{aligned} & \Pr_{(\theta_0, \sigma)} (|X - \theta_0| < |T(X) - \theta_0|) \\ & \geq \Pr_{(\theta_0, \sigma)} \left\{ \max \left(\theta_0 - \delta_1, \theta_0 - \frac{\eta}{3} \right) < X < \theta_0 \right\} \\ & \quad + \Pr_{(\theta_0, \sigma)} \left\{ \theta_0 < X < \min \left(\theta_0 + \delta_2, \theta_0 + \frac{\eta}{3} \right) \right\} \quad . \end{aligned}$$

It is now true that

$$\begin{aligned} & \Pr_{(\theta_0, \sigma)} \left\{ \max \left(\theta_0 - \delta_1, \theta_0 - \frac{\eta}{3} \right) < X < \theta_0 \right\} \\ & \quad + \Pr_{(\theta_0, \sigma)} \left\{ \theta_0 < X < \min \left(\theta_0 + \delta_2, \theta_0 + \frac{\eta}{3} \right) \right\} \\ & = \Pr_{(\theta_0, \sigma)} \left\{ \max \left(\theta_0 - \delta_1, \theta_0 - \frac{\eta}{3} \right) < X < \min \left(\theta_0 + \delta_2, \theta_0 + \frac{\eta}{3} \right) \right\} \\ & = \Pr_{(\theta_0, \sigma)} \left\{ \max \left(-\frac{\delta_1}{\sigma}, -\frac{\eta}{3\sigma} \right) < \frac{X - \theta_0}{\sigma} < \min \left(\frac{\delta_2}{\sigma}, \frac{\eta}{3\sigma} \right) \right\} \quad , \end{aligned}$$

and

$$\lim_{\sigma \rightarrow 0} \Pr_{(\theta_0, \sigma)} \left\{ \max \left(-\frac{\delta_1}{\sigma}, -\frac{\eta}{3\sigma} \right) < \frac{X - \theta_0}{\sigma} < \min \left(\frac{\delta_2}{\sigma}, \frac{\eta}{3\sigma} \right) \right\} = 1 \quad .$$

(Q.E.D)

CHAPTER 4. MULTIVARIATE SHRINKAGE CONSTRUCTIONS

4.1 Introduction

Let the p -dimensional vectors X and Y , with joint density depending on a parameter vector $\omega = (\theta_1, \dots, \theta_p, \eta_1, \dots, \eta_q)$, $\omega \in \Omega \subseteq R^{p+q}$, be estimators of $\theta = (\theta_1, \dots, \theta_p)$. X is closer than Y to θ with respect to the norm $\|\cdot\|_Q$ (or, in the terminology used below, X Pitman-dominates Y) if

$$\Pr_{\omega}(\|X - \theta\|_Q < \|Y - \theta\|_Q) > 1/2, \quad \forall \omega \in \Omega,$$

where, for given positive definite matrix Q , $\|X - \theta\|_Q^2$ is defined as $(X - \theta)'Q(X - \theta)$. This PCC is defined by Sen *et al* [32]. We consider the case of $Q = I$ in this chapter.

4.2 James-Stein Type Estimators for Pitman Domination

Consider, throughout this section, the vector of estimators $X = (X_1, \dots, X_p)'$ where $p \geq 2$, $X \sim N_p(\theta, \sigma^2 V)$, $\theta = (\theta_1, \dots, \theta_p)'$, and V is a known positive definite matrix. We assume that there exists a statistic S distributed independently of X as $\sigma^2 \chi_m^2/m$, where χ_m^2 is a chi-square random variable with m degrees of freedom. We consider construction of estimators Pitman-dominating X , estimators of unknown θ , in this section. The simplest case is that σ^2 is known and $V = I$.

For simplicity, we consider the vector of estimators $X = (X_1, \dots, X_p)'$ where $X \sim N_p(\theta, I)$ and $\theta = (\theta_1, \dots, \theta_p)'$. Let $\phi \equiv \phi(X) = (\phi(X_1), \dots, \phi(X_p))'$ and $T \equiv T(X) = X - \phi(X)$. Then we observe that

$$\begin{aligned} \|T - \theta\|^2 - \|X - \theta\|^2 &= ((X - \theta) - \phi)'((X - \theta) - \phi) - (X - \theta)'(X - \theta) \\ &= -2(X - \theta)\phi' + \phi'\phi . \end{aligned}$$

Thus

$$\Pr_{\theta}(\|T - \theta\| < \|X - \theta\|) = \Pr_{\theta}\left((X - \theta)\phi' > \frac{\phi'\phi}{2}\right)$$

Hence, the problem is now to find a function $\phi(X)$ satisfying

$$\Pr_{\theta}\left((X - \theta)\phi' > \frac{\phi'\phi}{2}\right) > \frac{1}{2}, \quad \forall \theta .$$

Let us take $\phi(X) = cX/X'X$ with constant $c > 0$ then

$$\begin{aligned} \|T - \theta\|^2 - \|X - \theta\|^2 &= -2(X - \theta)\phi' + \phi'\phi \\ &= -2c(X - \theta)X'/X'X + c^2/X'X . \end{aligned}$$

Thus

$$\begin{aligned} \Pr_{\theta}(\|T - \theta\| < \|X - \theta\|) &= \Pr_{\theta}\left((X - \theta)X' > \frac{c}{2}\right) \\ &= \Pr_{\theta}\left((X - \theta/2)'(X - \theta/2) > \frac{c}{2} + \frac{\theta'\theta}{4}\right) \\ &= \Pr_{\delta}\left(\chi_{p,\delta}^2 > \frac{c}{2} + \delta\right) \end{aligned}$$

where $\delta = \frac{\theta'\theta}{4}$ and $\chi_{p,\delta}^2$ is a non-central chi-square random variable with p degrees of freedom and non-centrality parameter δ .

James and Stein [13] showed that T , where

$$T = (1 - c/X'X)X$$

has smaller mean square error than X for all θ provided $0 < c < 2(p - 2)$ for $p \geq 3$. and has smallest mean square error if $c = p - 2$.

Efron [7], Rao, *et al.* [28], Keating, *et al.* [17], Sen, *et al.* [32, 33], Saleh, *et al.* [29] considered James-Stein type estimators under Pitman closeness criterion. It may be useful to review some of the key theorems developed by them.

In addition to X and T , define

$$\begin{aligned} T^+ &= (1 - c/X'X)^+ X, \quad c > 0 \\ &= T \cdot I(X'X > c), \quad c > 0, \end{aligned}$$

where $(1 - c/X'X)^+ = ((1 - c/X'X) \vee 0) = \max(1 - c/X'X, 0)$.

The following theorems compare T , T^+ , and X under the Pitman closeness criterion.

Theorem 4.1 (Sen, Kubokawa and Saleh) *For every $p \geq 2$, $T = (1 - c/X'X)X$ Pitman-dominates X if $0 < c < (p - 1)(3p - 1)/(2p)$.*

The proof is omitted since it is given by Sen *et al.* [32]. Theorem 4.1 is equivalent to stating that

$$P(T, X; \theta) = \Pr_{\delta} \left(\chi_{p, \delta}^2 > \frac{c}{2} + \delta \right) > \frac{1}{2} \quad \forall \theta \quad (\text{i.e., } \forall \delta),$$

where $\delta = \frac{\theta' \theta}{4}$ and $\chi_{p, \delta}^2$ is a non-central chi-square random variable with p degrees of freedom and non-centrality parameter δ .

Theorem 4.2 (Saleh and Sen) *For every $p \geq 2$, T^+ Pitman-dominates T .*

The proof is omitted since it is given by Saleh and Sen [29].

Let $\Theta^+ = \{\theta \in R^p : \theta > 0\}$ and, for any $X \in R^p$, let $X^+ = (X_1 \vee 0, \dots, X_p \vee 0)'$. Sen and Sengupta [33] compared the restricted maximum likelihood estimators X^+ with the corresponding James-Stein type estimators

$$T_R^+ = \left(1 - \frac{c_a(X)}{\|X^+\|^2}\right) X^+ ,$$

where

$$a(X) = \sum_{i=1}^p I(X_i > 0) ,$$

and

$$c_k = \text{median}(\chi_p^2) , \quad 2 \leq k \leq p ;$$

$$c_0 = c_1 = 0 .$$

Theorem 4.3 (Sen and Sengupta) *For every $p \geq 2$, T_R^+ Pitman-dominates X^+ .*

The proof is omitted since it is given by Sen and Sengupta [33].

Lemma 4.1 *Let $T_i = (1 - c_i/X'X)X$ for $i = 1, 2$, where $c_1 > c_2 > 0$. Then, for every $p \geq 2$,*

$$\begin{aligned} P(T_1, T_2; \theta) &= \Pr_{\theta} (\|T_1 - \theta\| < \|T_2 - \theta\|) \\ &= \Pr_{\theta} \left(\chi_{p,\delta}^2 > \frac{c_1 + c_2}{2} + \delta \right) \end{aligned}$$

where $\delta = \frac{\theta'\theta}{4}$ and $\chi_{p,\delta}^2$ is a non-central chi-square random variable with p degrees of freedom and non-centrality parameter δ .

Proof: We have that

$$\begin{aligned}
P(T_1, T_2; \theta) &= \Pr_{\theta} \left((T_1 - \theta)'(T_1 - \theta) < (T_2 - \theta)'(T_2 - \theta) \right) \\
&= \Pr_{\theta} \left(\{(X - \theta) - c_1 X/X'X\}^2 < \{(X - \theta) - c_2 X/X'X\}^2 \right) \\
&= \Pr_{\theta} \left((c_1 - c_2)(X - \theta)'X > \frac{c_1^2 - c_2^2}{2} \right) \\
&= \Pr_{\theta} \left((X - \theta/2)'(X - \theta/2) > \frac{c_1 + c_2}{2} + \frac{\theta'\theta}{4} \right) \\
&= \Pr_{\delta} \left(\chi_{p,\delta}^2 > \frac{c_1 + c_2}{2} + \delta \right) \quad (\text{Q.E.D.})
\end{aligned}$$

The following theorem shows that the dominating function of Theorem 4.1 itself can be dominated.

Theorem 4.4 Let $T_i = (1 - c_i/X'X)X$ for $i = 1, 2$, where $c_1 > c_2 > 0$. Then, for every $p \geq 2$ and given c_2 , $0 < c_2 < (p-1)(3p+1)/(4p)$, T_1 Pitman-dominates T_2 provided

$$c_2 < c_1 < \frac{(p-1)(3p+1)}{2p}.$$

Proof: From Lemma 4.1, we have that

$$P(T_1, T_2; \theta) = \Pr_{\theta} \left(\chi_{p,\delta}^2 > \frac{c_1 + c_2}{2} + \delta \right).$$

Now, in view of Theorem 4.1, if we have

$$0 < c_1 + c_2 < \frac{(p-1)(3p+1)}{2p}$$

then T_1 Pitman-dominates T_2 . Thus, for given c_2 , $0 < c_2 < (p-1)(3p+1)/(4p)$, if we take c_1 such that

$$c_2 < c_1 < \frac{(p-1)(3p+1)}{2p} - c_2$$

then T_1 Pitman-dominates T_2 . (Q.E.D.)

Now define

$$T(b) = (1 - b/X'X)^+ X$$

where $0 < b < (p-1)(3p+1)/(2p)$, and define $T_M \equiv T(m_p)$ where m_p is the median of chi-square random variable with degrees of freedom p . And further define

$$C = \{T(b) : m_p < b \leq (p-1)(3p+1)/(2p)\}.$$

Then we can state the following theorem which is proved by Sen and Sengupta [33].

Theorem 4.5 (Sen and Sengupta) *For every $p \geq 2$, T_M Pitman-closest than any estimator in C .*

Theorem 4.6 *For every $p \geq 2$ and $a \in R$, define*

$$T_{c,a} \equiv T_{c,a}(X) = \left\{ 1 - \frac{c}{(X - a\mathbf{1})'(X - a\mathbf{1})} \right\} (X - a\mathbf{1}) + a\mathbf{1}$$

, where $\mathbf{1} = (1, \dots, 1)'$. Then, $T_{c,a}$ Pitman-dominates X if $0 < c < (p-1)(3p+1)/(2p)$.

Proof: Since $X \sim N_p(\theta, I)$, $Y = X - a\mathbf{1} \sim N_p(\mu, I)$ where $\mu = \theta - a\mathbf{1}$. Then the situation is now exactly equal to Theorem 4.1. (Q.E.D.)

Theorem 4.6 states that shrinkage construction need not always be toward the

origin in location problems, the construction may be toward any vector which has same real numbers as its element.

Note 4.1: The case of $X \sim N_p(\theta, \sigma^2 I)$ with known σ^2 can be dealt within this section since $Y \equiv X/\sigma \sim N_p(\theta, I)$.

Note 4.2: Consider the general setup stated in Section 3.2 and generalized Pitman closeness criterion with loss function $L(X - \theta) = (X - \theta)'Q(X - \theta)$ for a given positive definite matrix Q . Stein [34] proposed Stein estimators of the form

$$T = X - c S \|X\|_{Q,V}^{-2} Q^{-1} V^{-1} X ,$$

where c is a nonnegative value, $\|X\|_{Q,V}^2 = X'V^{-1}Q^{-1}V^{-1}X$, V is a covariance matrix, and mS/σ^2 is independent of X and distributed with chi-square with m degrees of freedom.

4.3 Multivariate Shrinkage Constructions for Generalized Pitman Domination

Let the p -dimensional vectors X and Y , with joint density depending on a parameter vector $\omega = (\theta_1, \dots, \theta_p, \eta_1, \dots, \eta_q)$, $\omega \in \Omega \subseteq R^{p+q}$, be estimators of $\theta = (\theta_1, \dots, \theta_p)$. X is closer than Y to θ with respect to the loss function $L(\cdot, \cdot)$ (or, in the terminology used below, X Pitman-dominates Y) if

$$\Pr_{\omega}(L(X, \theta) < L(Y, \theta)) > 1/2 , \quad \forall \omega \in \Omega . \quad (4.1)$$

This criterion is defined by Rao [28] and Sen [32]), and is called the Generalized Pitman Closeness Criterion (GPCC). In the current chapter, we consider additive loss

functions of the form $L(x, \theta) = \sum_{i=1}^P L_i(x_i, \theta_i)$; here $L_i(x_i, \theta_i) = h_i(x_i - \theta_i)$ where $h_i(y) = r_i(y)$ on $[0, +\infty)$ and $s_i(-y)$ on $(-\infty, 0]$, with $r_i(y)$ and $s_i(y)$ continuous increasing on $[0, +\infty)$ and $r_i(0) = s_i(0) = 0$.

One point made in the present section is that the Pitman closeness criterion makes graphic that the Stein effect manifests itself more readily in higher dimensions. That point is made especially naturally in the context of Theorem 2.8 in Chapter 2 which studies shrinkage constructions for univariate Pitman domination in a rather general setting. Theorem 4.7 links the univariate considerations of Theorem 2.8 to GPCC under the above additive loss functions, and is followed by examples. It can be noted that David and Salem [5] and the results of Chapter 2 essentially use the notion of “crossing point,” as it appears in Rao *et al* [28], but with emphasis on *construction*, rather than *comparison*, of estimators.

We begin with the following preliminary observation. In the above GPCC setting, with $X = (X_1, \dots, X_p)$ and $T = (T_1, X_2, \dots, X_p)$,

$$\begin{aligned} \Pr_{\omega}(L(T, \theta) < L(X, \theta)) &> 1/2, \forall \omega \in \Omega \\ \Rightarrow \Pr_{\omega}(L_1(T_1, \theta_1) < L_1(X_1, \theta_1)) &> 1/2, \forall \omega \in \Omega. \end{aligned} \quad (4.2)$$

This observation, applied in the context of Theorem 2.8, leads to the following theorem.

Theorem 4.7 *Let the p -dimensional vector $X = (X_1, \dots, X_p)$, with density depending on a parameter vector $\omega = (\theta_1, \dots, \theta_p, \eta_1, \dots, \eta_q)$, estimate $\theta = (\theta_1, \dots, \theta_p)$. Suppose that the domain of θ_1 is an interval $I \equiv (a_1, a_2)$, and that the marginal*

density of X_1 depends only on θ_1 ; further suppose that this density is supported by I for all $\theta_1 \in I$, and that X_1 is median-unbiased for θ_1 . Then there exists an estimator of θ Pitman-dominating X with respect to all additive loss functions of type L .

Proof: Using (4.2) and the conditions of the theorem, we have that

$$\begin{aligned} \Pr_{\omega}(L(T, \theta) < L(X, \theta)) &> 1/2, \forall \omega \in \Omega \\ \Rightarrow \Pr_{\omega}(L_1(T_1, \theta_1) < L_1(X_1, \theta_1)) &> 1/2, \forall \omega \in \Omega \\ \Rightarrow \Pr_{\theta_1}(L_1(T_1, \theta_1) < L_1(X_1, \theta_1)) &> 1/2, \forall \theta_1 \in I. \end{aligned}$$

Now take T_1 Pitman-dominating X_1 as an estimator of θ_1 as constructed in Theorem 2.8. \square

Example 4.1 (Here $(a_1, a_2) = (-\infty, +\infty)$ and $\omega = (\theta_1, \dots, \theta_p)$.) Consider a multivariate location family of density $g(x_1 - \theta_1, \dots, x_p - \theta_p)$ pertaining to a vector $X = (X_1, \dots, X_p)$ of estimators of $\theta = (\theta_1, \dots, \theta_p)$. If X_1 is distributed symmetrically about θ_1 , then there exists an estimator of θ Pitman-dominating X with respect to all additive loss functions of type L . \square

Example 4.2 (Here $(a_1, a_2) = (0, +\infty)$ and $q = p(p-1)/2$.) Suppose that, for some positive integer p , $X = (X_1, \dots, X_p)$ has p -variate normal distribution with all parameters unknown. Consider the vector $S = (S_1^2, \dots, S_p^2)$ of estimators of $\sigma^2 = (\sigma_1^2, \dots, \sigma_p^2)$ with S_1^2 median-unbiased for σ_1^2 . Then there exists an estimator of σ^2 Pitman-dominating S with respect to all additive loss functions of type L . \square

Observation (4.2) also is useful when applied in conjunction with univariate

constructions not necessarily involving median-unbiasedness; one such instance is that of the evident extension of Example 2.4 in Chapter 2 beyond PCC, leading to the following example.

Example 4.3 Let $X \sim N(\theta, \sigma^2)$, where both θ and σ^2 are unknown. Consider the vector $(\bar{X}/S, \bar{X})$ of estimators of $(\theta/\sigma, \theta)$, where \bar{X} is the sample mean and S^2 is the unbiased estimator of σ^2 . Then there exists an estimator of $(\theta/\sigma, \theta)$ Pitman-dominating $(\bar{X}/S, \bar{X})$ with respect to all additive loss functions of type L . \square

The following example combines the ideas of Theorem 3.1 and Theorem 4.7.

Example 4.4 Consider a multivariate location-scale family of density $f(\frac{x_1 - \theta_1}{\sigma_1}, \dots, \frac{x_p - \theta_p}{\sigma_p})$ pertaining to a vector $X = (X_1, \dots, X_p)$ of estimators of $\theta = (\theta_1, \dots, \theta_p)$. If X_1 is distributed symmetrically about θ_1 and σ_1 is bounded below, then there exists an estimator of θ Pitman-dominating X with respect to all additive loss functions of type L . \square

CHAPTER 5. TRIPLES OF INTRANSITIVE ESTIMATORS

5.1 Introduction

We consider univariate version of GPCC in this chapter as follows: Let X and Y , with joint density depending on a parameter vector $\omega = (\theta, \eta)$, be estimators of θ . X is closer than Y to θ with respect to the loss function $L(\cdot, \cdot)$ (or, in the notation used below, $X \triangleright Y$) if

$$\Pr_{\omega}(L(X, \theta) < L(Y, \theta)) > 1/2, \quad \forall \omega \in \Omega. \quad (5.1)$$

After Pitman [26] noted that PCC is intransitive, David [4] defined stochastic transitivity and showed the existence of stochastic intransitive triples. He considered stochastic transitivity in the course of investigating ranking problems for the method of paired comparisons. Blyth [2] further discussed stochastic intransitivity problems with discrete random variables. It is possible to interpret that the works of David [4] and Blyth [2] are related to Pitman-intransitive for a degenerate parameter space. David and Salem [5] constructed intransitive triples of estimators of a Laplace location parameter, each member of the triple dominating a single observation in the sense of PCC.

The present chapter is devoted to constructing a class of Pitman-intransitive triples $X, M(X), T(X)$ with $X \triangleright M(X)$, $T(X) \triangleright X$, and $T(X) \not\triangleright M(X)$ for the

location parameter case and scale parameter case; here the first domination and second domination respectively are based on the two lines of research cited above. The discussion is in terms of GPC'C, involving loss functions of the form $L(x, \theta) = h(x - \theta)$, where $h(y) = r(y)$ on $[0, +\infty)$ and $s(-y)$ on $(-\infty, 0]$, with $r(y)$ and $s(y)$ continuous increasing on $[0, +\infty)$ and $r(0) = s(0) = 0$.

5.2 Triples of Intransitive Estimators for Location Parameters

Theorem 2.8 with $c = 0$, $a_1 = -\infty$ and $a_2 = +\infty$, and the results of Ghosh and Sen [10], Nayak [22], and Kubokawa [19], lead to the following theorem, whose geometric basis is illustrated by Figure 5.1.

Theorem 5.1 *Let X , median-unbiased for a location parameter θ , $\theta \in I = (-\infty, +\infty)$, have density $f(x; \theta)$ with support I for all θ . Let, for any fixed $\theta = \theta_0 \in (0, +\infty)$, $M(X)$ be any estimator of θ of the form $M(X) = X + b$, $b < 0$, satisfying, for $\theta_0 \leq x \leq \lambda(\theta_0)$,*

$$\max(x + \lambda^{-1}(\theta_0) - \theta_0, \lambda^{-1}(x), x + \theta_0 - \lambda(\theta_0)) \leq M(x) < x . \quad (5.2)$$

Let, for given $x_0 \in (0, \theta_0)$, $T(X)$ be any estimator of θ which is continuous and

$$x < T(x) < \lambda^{-1}(x) , \quad x < 0 \quad (5.3a)$$

$$T(0) = 0 , \quad (5.3b)$$

$$\max(\lambda^{-1}(x), M(x)) < T(x) < x , \quad 0 < x < x_0 \quad (5.3c)$$

$$T(x_0) = M(x_0) , \quad (5.3d)$$

$$\lambda^{-1}(x) < T(x) < M(x) \ , \ x_0 < x < M^{-1}(\theta_0) \quad (5.3e)$$

$$T(x) = M(x) \ , \ x = M^{-1}(\theta_0) \quad (5.3f)$$

$$M(x) < T(x) < x \ , \ M^{-1}(\theta_0) < x \leq \lambda(\theta_0) \quad (5.3g)$$

$$\max(\lambda^{-1}(x), M(x)) < T(x) < x \ , \ x > \lambda(\theta_0) \ . \quad (5.3h)$$

Then $X \triangleright M(X)$, $T(X) \triangleright X$, and $T(X) \not\triangleright M(X)$; in other words,

$$\Pr_{\theta}(L(X, \theta) < L(M(X), \theta)) > 1/2 \ , \ \forall \theta \in I \ , \quad (5.4a)$$

$$\Pr_{\theta}(L(T(X), \theta) < L(X, \theta)) > 1/2 \ , \ \forall \theta \in I \ , \quad (5.4b)$$

$$\Pr_{\theta}(L(M(X), \theta) < L(T(X), \theta)) > 1/2 \ , \ \theta = \theta_0 \ . \quad (5.4c)$$

Proof: Essentially as in Ghosh and Sen [10], Nayak [22], and Kubokawa [19]. $X \triangleright M(X)$. Also it is true that $T(X) \triangleright X$ since $T(X)$ satisfying (5.2) and (5.3) also satisfies condition (2.49) in Theorem 2.8.

It remains to show that (5.4c) holds. In view of (5.2), it is observed that

$$\max(\lambda^{-1}(\theta_0), 2\theta_0 - \lambda(\theta_0)) \leq M(\theta_0) < \theta_0 \quad (5.5)$$

and

$$\max(\lambda(\theta_0) + \lambda^{-1}(\theta_0) - \theta_0, \theta_0) \leq M(\lambda(\theta_0)) < \lambda(\theta_0) \ . \quad (5.6)$$

(5.5) implies

$$\theta_0 < M^{-1}(\theta_0) \quad (5.7)$$

and (5.6) implies

$$M^{-1}(\theta_0) < \lambda(\theta_0) \ . \quad (5.8)$$

Hence, in view of (5.7) and (5.8), it is true that

$$\theta_0 < M^{-1}(\theta_0) < \lambda(\theta_0) . \quad (5.9)$$

For $x_0 < x < M^{-1}(\theta_0)$, it is clear that $M(x) < \theta_0$, hence, in view of (5.3e), we have

$$L(M(x), \theta_0) < L(T(x), \theta_0) . \quad (5.10)$$

For $M^{-1}(\theta_0) < x \leq \lambda(\theta_0)$, it is clear that $\theta_0 < M(x)$, hence, in view of (5.3g), we have

$$L(M(x), \theta_0) < L(T(x), \theta_0) . \quad (5.11)$$

For $x > \lambda(\theta_0)$, it is clear that $\lambda^{-1}(x) > \theta_0$ and, in view of (5.8), it is true that

$$M(x) > M(\lambda(\theta_0)) > M(M^{-1}(\theta_0)) = \theta_0 ,$$

hence,

$$\max(M(x), \lambda^{-1}(x)) > \theta_0 . \quad (5.12)$$

Thus, in view of (5.3h) and (5.12), we have

$$L(M(x), \theta_0) < L(T(x), \theta_0) . \quad (5.13)$$

Therefore, in view of (5.3f), (5.10), (5.11), and (5.13),

$$x > x_0$$

implies

$$L(M(x), \theta_0) \leq L(T(x), \theta_0)$$

with equality holding only when $x = M^{-1}(\theta)$.

Finally, we have, for $\theta = \theta_0 \in (0, +\infty)$,

$$\begin{aligned} & \Pr_{\theta}(L(M(X), \theta) < L(T(X), \theta)) \\ & \geq \Pr_{\theta}(X > x_0) \\ & > \Pr_{\theta}(X > \theta) = 1/2 . \quad \square \end{aligned}$$

Note 5.1: For any fixed $\theta = \theta_0 \in (-\infty, 0)$, a similar argument holds by defining $M(X)$ to be any equivariant estimator of θ of the form $M(X) = X + b$, $b > 0$, satisfying, for $\lambda(\theta_0) \leq x \leq \theta_0$,

$$x < M(x) \leq \min(x + \lambda^{-1}(\theta_0) - \theta_0, \lambda^{-1}(x), x + \theta_0 - \lambda(\theta_0)) .$$

Note 5.2: In keeping with the fact that shrinkage need not be constructed with $c = 0$, the construction can be done with arbitrary $c \in (-\infty, +\infty)$.

Example 5.1 Let X be a single observation from the density

$$f(x; \theta) = 2^{-1} e^{-|x-\theta|} , \quad x \in (-\infty, +\infty) ,$$

where θ is real valued unknown location parameter. It is noted that X is a median-unbiased estimator of θ . Without loss of generality, we assume $\theta > 0$. It is easy to see that $\lambda^*(\theta) = \theta - \ln(1 - e^{-\theta})$ where $\lambda^*(\theta)$ satisfies

$$\int_0^{\lambda^*(\theta)} f(x; \theta) dx = 1/2 , \quad 0 < \theta < +\infty .$$

It is easy to see that $\lambda^*(x)$ is convex and has a unique minimum value $2\ln 2$ at $x = \ln 2$. Now take $\lambda(\theta)$ satisfying

$$\lambda(\theta) = -\theta - \ln(1 - e^{\theta}) , \quad \theta < -\ln 2$$

$$\lambda(\theta) = 2\theta \quad , \quad -\ln 2 \leq \theta \leq \ln 2$$

$$\lambda(\theta) = \theta - \ln(1 - e^{-\theta}) \quad , \quad \theta > \ln 2 \quad .$$

Then $\lambda(\theta)$ satisfies (2.48). Now take $M(X)$ and $T(X)$ satisfying (5.2) and (5.3), respectively. Then, in view of Theorem 5.1, $X \triangleright M(X)$, $T(X) \triangleright X$, and $T(X) \not\triangleright M(X)$. \square

Example 5.2 Let Y_1, \dots, Y_n be iid random variables having the normal density with unknown mean θ and unit variance. Let X be the sample mean (i.e., $X \equiv \bar{Y} = \sum_{i=1}^n Y_i/n$). It is noted that X is a median-unbiased estimator of θ . In view of Theorem 2.8, we can construct a function $\lambda^{-1}(x)$ satisfying (2.48). Now take $M(X), T(X)$ satisfying (5.2) and (5.3), respectively. Then, in view of Theorem 5.1, $X \triangleright M(X)$, $T(X) \triangleright X$, and $T(X) \not\triangleright M(X)$. An example of triples of intransitive estimators of θ with $c = 0$ is displayed in Figure 5.1 by applying Theorem 5.1. \square

5.3 Triples of Intransitive Estimators for Scale Parameters

Theorem 2.8 with $c = 1$, $a_1 = 0$ and $a_2 = +\infty$, and the results of Ghosh and Sen [10], Nayak [22], and Kubokawa [19], lead to the following theorem, whose geometric basis is illustrated by Figure 5.2.

Theorem 5.2 Let X , median-unbiased for a scale parameter σ , $\sigma \in I = (0, +\infty)$, have density $f(x; \sigma)$ with support I for all σ . Let, for any fixed $\sigma = \sigma_0 \in (c, +\infty)$, $M(X)$ be any estimator of σ of the form $M(X) = bX$, $0 < b < 1$,

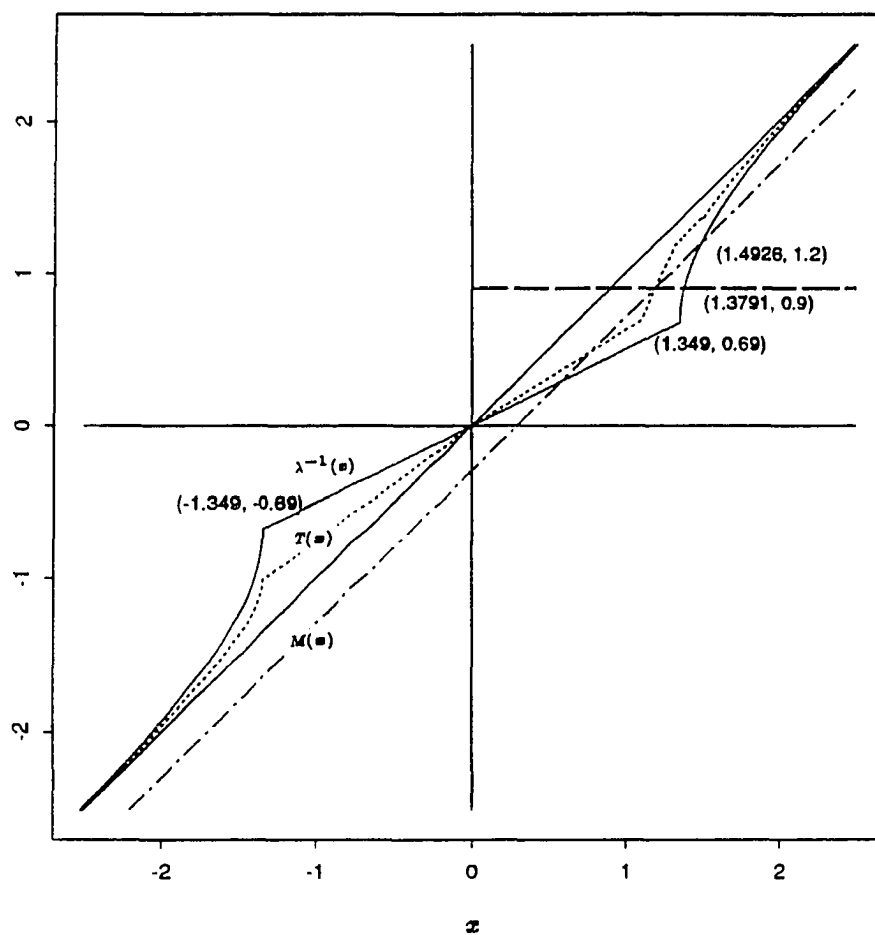


Figure 5.1: Triples of intransitive estimators of θ for $N(\theta, 1)$ with $c = 0$

satisfying, for $\sigma_0 \leq x \leq \lambda(\sigma_0)$,

$$\max(\lambda^{-1}(\sigma_0)x/\sigma_0, \lambda^{-1}(x), \sigma_0 x/\lambda(\sigma_0)) \leq M(x) < x . \quad (5.14)$$

Let, for given $x_0 \in (c, \sigma_0)$, $T(X)$ be any estimator of σ which is continuous and

$$x < T(x) < \lambda^{-1}(x) , \quad 0 < x < c \quad (5.15a)$$

$$T(c) = c , \quad (5.15b)$$

$$\max(\lambda^{-1}(x), M(x)) < T(x) < x , \quad c < x < x_0 \quad (5.15c)$$

$$T(x_0) = M(x_0) , \quad (5.15d)$$

$$\lambda^{-1}(x) < T(x) < M(x) , \quad x_0 < x < M^{-1}(\sigma_0) \quad (5.15e)$$

$$T(x) = M(x) , \quad x = M^{-1}(\sigma_0) \quad (5.15f)$$

$$M(x) < T(x) < x , \quad M^{-1}(\sigma_0) < x \leq \lambda(\sigma_0) \quad (5.15g)$$

$$\max(\lambda^{-1}(x), M(x)) < T(x) < x , \quad x > \lambda(\sigma_0) . \quad (5.15h)$$

Then $X \triangleright M(X)$, $T(X) \triangleright X$, and $T(X) \not\triangleright M(X)$; in other words,

$$\Pr_{\sigma}(L(X, \sigma) < L(M(X), \sigma)) > 1/2 , \quad \forall \sigma \in I ,$$

$$\Pr_{\sigma}(L(T(X), \sigma) < L(X, \sigma)) > 1/2 , \quad \forall \sigma \in I ,$$

$$\Pr_{\sigma}(L(M(X), \sigma) < L(T(X), \sigma)) > 1/2 , \quad \sigma = \sigma_0 .$$

The proof is omitted since it essentially follows the proof of Theorem 5.1.

Note 5.3: For any fixed $\sigma = \sigma_0 \in (0, c)$, $M(X)$ be any equivariant estimator of σ of the form $M(X) = bX$, $b > 1$, satisfying, for $\lambda(\sigma_0) \leq x \leq \sigma_0$,

$$x < M(x) \leq \min(\lambda^{-1}(\sigma_0)x/\sigma_0, \lambda^{-1}(x), \sigma_0 x/\lambda(\sigma_0)) .$$

Example 5.3 Let Y_1, \dots, Y_n be iid random variables having the normal density with unknown mean θ and unknown variance σ^2 . Let S^2 be the median-unbiased estimator of σ^2 (i.e., $S^2 = \sum_{i=1}^n (Y_i - \bar{Y})^2 / (n-1)$). It is observed that

$$\Pr_{\sigma}(0 < S^2 < \sigma^2) = \Pr_{\sigma}(0 < \frac{(n-1)S^2}{\sigma^2} < n-1) > \frac{1}{2} .$$

Thus, there exist $k_n, 0 < k_n < 1$, such that

$$\Pr_{\sigma}(0 < \frac{S^2}{k_n} < \sigma^2) = \frac{1}{2} .$$

Therefore, $S^2/k_n \equiv X$ is a median-unbiased estimator of σ^2 . Now it is clear that, for any $c > 0$, there exist a continuous increasing function $\lambda(\sigma^2)$ such that

$$\Pr_{\sigma}(c < X < \lambda(\sigma^2)) = \frac{1}{2} , \lambda(\sigma^2) > \sigma^2 > c ,$$

$$\lambda(c) = c ,$$

$$\Pr_{\sigma}(\lambda(\sigma^2) < X < c) = \frac{1}{2} , \lambda(\sigma^2) < \sigma^2 < c .$$

Now take $M(X), T(X)$ satisfying (5.14) and (5.15), respectively. Then, in view of Theorem 5.2, $X \triangleright M(X)$, $T(X) \triangleright X$, and $T(X) \not\triangleright M(X)$. An example of triples of intransitive estimators of σ^2 with $c = 1$ is displayed in Figure 5.2 by applying Theorem 5.2. \square

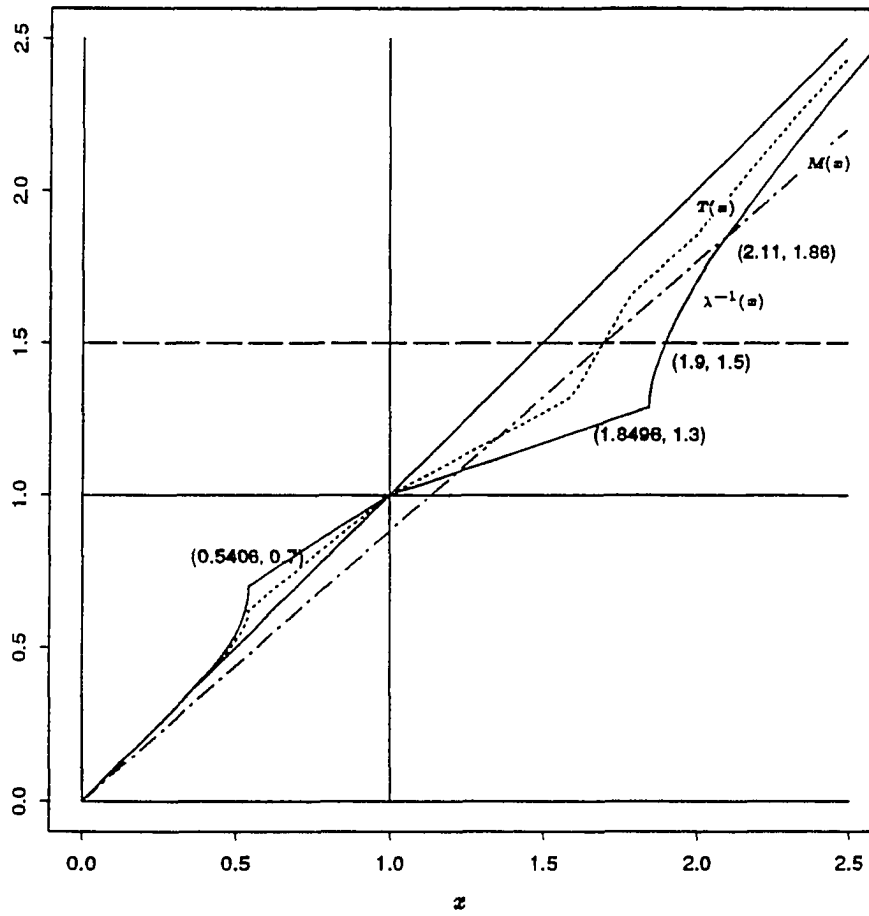


Figure 5.2: Triples of intransitive estimators of σ^2 for $N(\theta, \sigma^2)$ when $n = 11$ with $c = 1$

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ACKNOWLEDGMENTS

I wish to express my sincere thanks to my advisor Prof. Herbert T. David for his kind guidance and many invaluable suggestions throughout this research.

I would like to acknowledge as well the help and being patient of my wife Mijong, my son Insung, and my daughter Suhyun.

I dedicate this dissertation to my parents, Euljun Yoo and Myoungsun Jin.